推定理論を用いた量子測定の誤差と擾乱の不確定性関係

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PRA **84**, 042121 (2011).
arXiv:1106.2526 (2011).

Outline

- Review of uncertainty relations.
- Information-theoretic formulation of error and disturbance by using quantum estimation theory (QET).
- Uncertainty relation between error and disturbance.

Review of Uncertainty Relation

- Heisenberg's γ-ray microscope
- Kennard-Robertson's inequality
- Arthurs-Goodman's inequality
- Ozawa's inequality

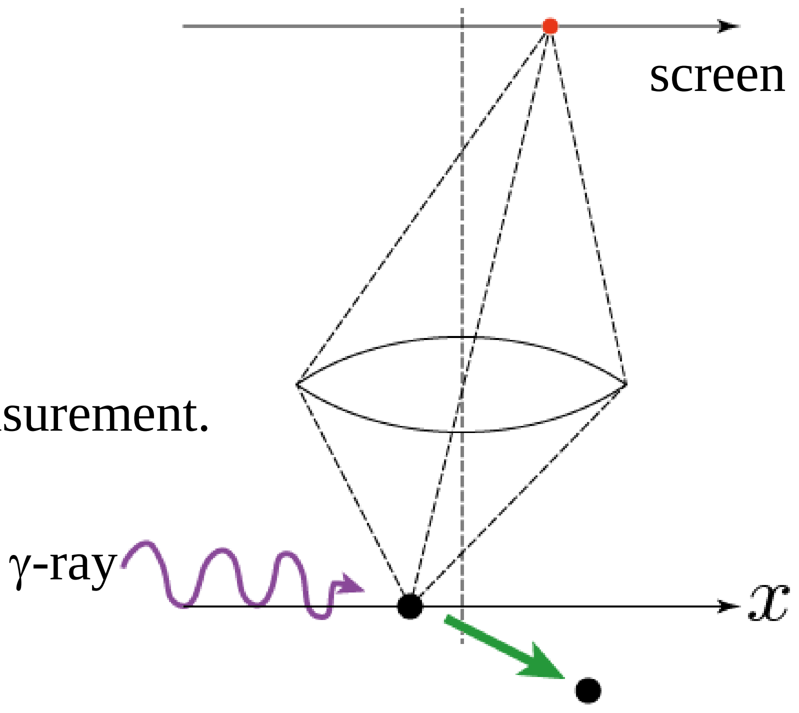
Heisenberg's γ−ray microscope

W. Heisenberg, Z. Phys. 43, 172 (1927).

- Semi-classical model of position measurement by using γ-ray.
  - Error of position: $\delta x \sim \lambda$
  - Disturbance to momentum: $\delta p_x \sim h/\lambda$

$$\Rightarrow \quad \delta x \, \delta p_x \gtrsim h$$

  - Existence of trade-off relations between error and disturbance is pointed out.
  - Called complementarity in quantum measurement.



- This argument depend on the specific model.
  How about in general quantum measurement?

Kennard–Robertson's inequality

For all observables $\hat{X}, \hat{Y}$

E. H. Kennard, Z. Phys. 44, 326 (1927).
H. P. Robertson, Phys. Rev. 34, 163 (1929).

$$\Delta X \Delta Y \geq \frac{1}{2} |\langle [\hat{X}, \hat{Y}] \rangle|$$

$$(\Delta X)^2 := \langle \hat{X}^2 \rangle - \langle \hat{X} \rangle^2$$: fluctuation of $\hat{X}$

$$\langle \hat{X} \rangle := \mathrm{Tr}[\hat{\rho} \hat{X}]$$

- Trade-off relation about fluctuation of observables.

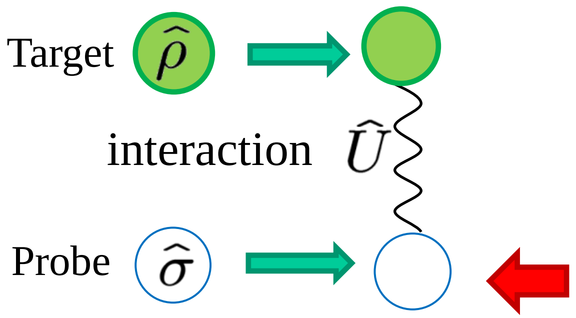
  Noncommuting observables do not have fixed values simultaneously.
  Called indeterminacy of quantum state.

- Depend only on quantum state. Independent of measurement process.

  Not the mathematical proof of Heisenberg's uncertainty relation.
  Complementarity and indeterminacy are different concept.

Indirect Measurement and Unbiased Measurement

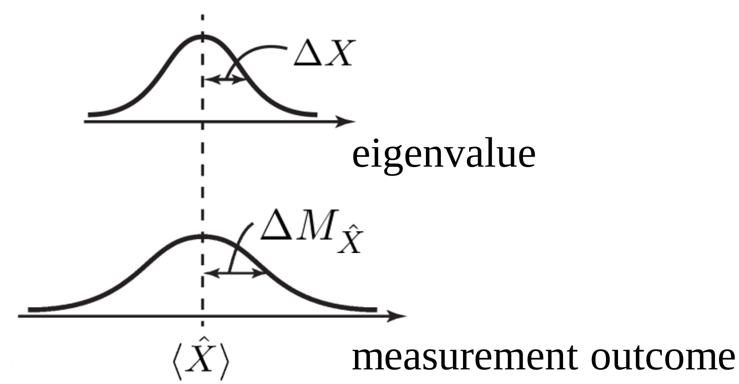
- Indirect measurement



$\boldsymbol{P} = \{\hat{P}_i\}$: Projection measurement of $\hat{M}_{\hat{X}}$

- Unbiased measurement

  The expectation values of observable
  and measurement outcome are equivalent.



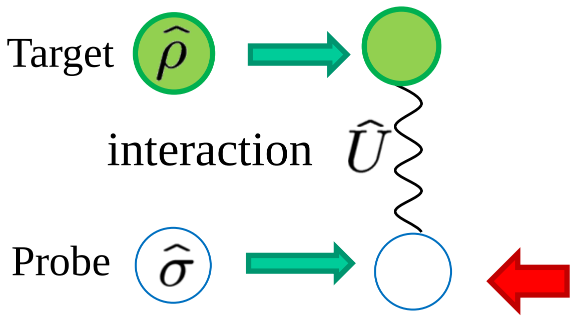
$$\mathrm{Tr}[\hat{\rho}_{\mathrm{tot}} \hat{M}_{\hat{X}}] = \mathrm{Tr}[\hat{\rho} \hat{X}]$$

We can know $\langle \hat{X} \rangle$ by measurement outcomes.

Arthurs–Goodman's inequality

E. Arthurs and M. S. Goodman, PRL **60**, 2447 (1988).

- Simultaneous unbiased measurement of $\hat{X}$ and $\hat{Y}$



$\boldsymbol{P} = \{\hat{P}_i\}$: Projection measurement of $\hat{M}_{\hat{X}}, \hat{M}_{\hat{Y}}$

$[\hat{M}_{\hat{X}}, \hat{M}_{\hat{Y}}] = 0$: condition of simultaneous measurement

$\mathrm{Tr}[\hat{\rho}_{\mathrm{tot}} \hat{M}_{\hat{X}}] = \mathrm{Tr}[\hat{\rho}\hat{X}]$

$\mathrm{Tr}[\hat{\rho}_{\mathrm{tot}} \hat{M}_{\hat{Y}}] = \mathrm{Tr}[\hat{\rho}\hat{Y}]$: conditions of unbiased measurement

- Fluctuation of observables

$$\Delta X \Delta Y \geq \frac{1}{2} |\langle [\hat{X}, \hat{Y}] \rangle|$$

- measurement error

$$(\Delta N_{\hat{X}})^2 := (\Delta M_{\hat{X}})^2 - (\Delta X)^2$$

$$\Delta N_{\hat{X}} \Delta N_{\hat{Y}} \geq \frac{1}{2} |\langle [\hat{X}, \hat{Y}] \rangle|$$

- fluctuation of measurement outcomes

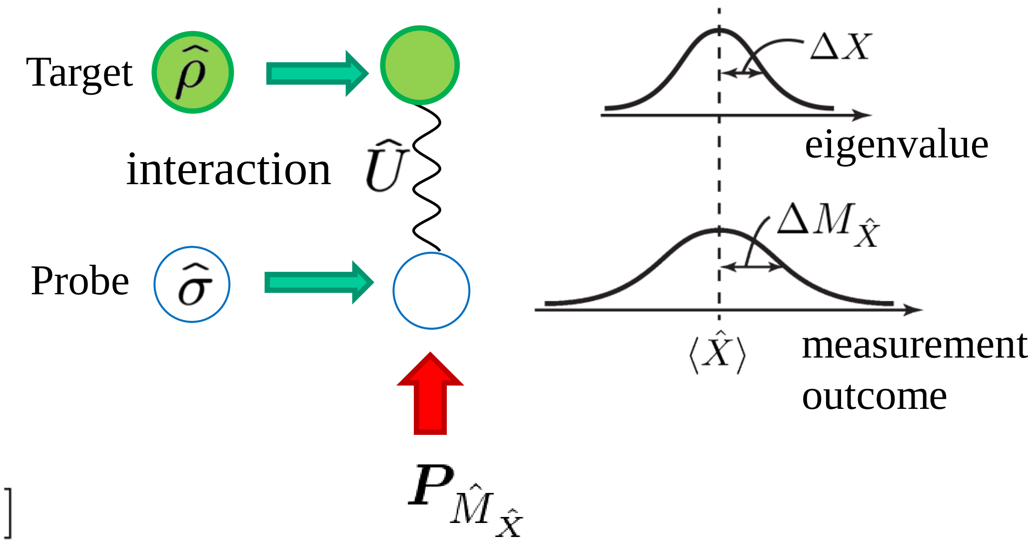
$$\Delta M_{\hat{X}} \Delta M_{\hat{Y}} \geq |\langle [\hat{X}, \hat{Y}] \rangle|$$

Ozawa's inequality

M. Ozawa, Phys. Lett. A **320**, 367 (2004).

- Error in unbiased measurement by Arthurs and Goodman

$$\varepsilon'(\hat{X})^2 := (\Delta M_{\hat{X}})^2 - (\Delta X)^2$$
$$= \mathrm{Tr}[(\hat{\rho}\otimes\hat{\sigma})\hat{N}_{\hat{X}}^2]$$
$$\hat{N}_{\hat{X}} := \hat{U}^\dagger \hat{M}_{\hat{X}} \hat{U} - \hat{X}$$



- Definition of error by Ozawa.

For all indirect measurement

$$\varepsilon'(\hat{X})^2 := \mathrm{Tr}[(\hat{\rho}\otimes\hat{\sigma})\hat{N}_{\hat{X}}^2]$$

- Ozawa's inequality

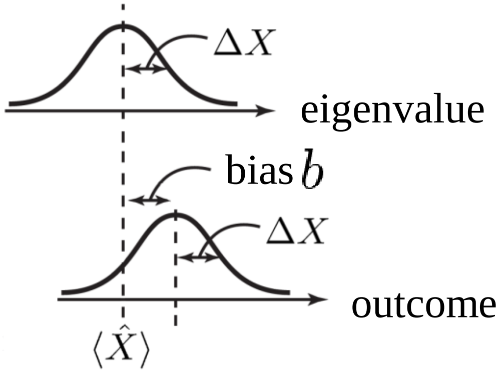
$$\varepsilon'(\hat{X})\varepsilon'(\hat{Y}) + \varepsilon'(\hat{X})\Delta\hat{Y} + \Delta\hat{X}\varepsilon'(\hat{Y}) \geq \frac{1}{2}|\langle[\hat{X},\hat{Y}]\rangle|$$

- Even if $\varepsilon'(\hat{X}) = 0$, from the definition of error, $\varepsilon'(\hat{Y}) < \infty$. Therefore,

$$\varepsilon'(\hat{X})\varepsilon'(\hat{Y}) \not\geq \frac{1}{2}|\langle[\hat{X},\hat{Y}]\rangle|$$

Comments on Ozawa's inequality

- Problems on definition of error in measurement
  - Projective measurement (precise measurement), and the outcomes are shifted: $\varepsilon'(\hat{X}) = b$

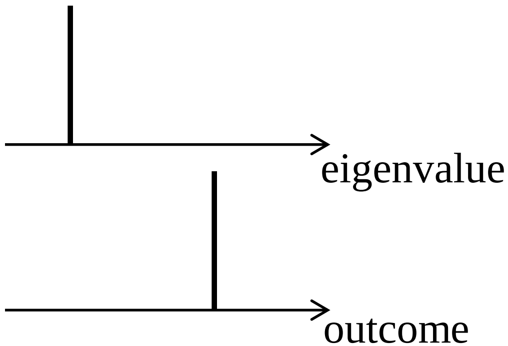


  - Projective measurement, and the outcomes are scaled by factor a : $\varepsilon'(\hat{X}) = (a-1)\sqrt{\langle \hat{X}^2 \rangle}$
  - Measurement outcome is constant k (no information) :

$$\varepsilon'(\hat{X}) = \sqrt{k^2 - 2k\langle \hat{X} \rangle + \langle \hat{X}^2 \rangle}$$

if $k = \langle \hat{X} \rangle$ and X does not fluctuate :

$$\varepsilon'(\hat{X}) = 0$$



Therefore, the error defined by Ozawa and Ozawa's inequality gives unfavorable results.

Summary of Review

- Heisenberg's microscope (complementarity)
  - Depend on specific model.
  - Semi-classical analysis
- Kennard-Robertson's inequality (indeterminacy)
  - Conceptually different trade-off relation
- Arthurs-Goodman's inequality (complementarity)
  - Only unbiased measurement
  - Disturbance is not discussed
- Ozawa's inequality (complementarity)
  - A generalization of Arthurs-Goodman's ineq.
  - Unreasonable definition of measurement error and disturbance

What trade-off relation between error and disturbance exists?
⇒ We give a resolution by using quantum estimation theory.

Information-theoretic Formulation of Error in Quantum Measurement

- Three types of error in quantum estimation theory
  - fluctuation of observable
  - measurement error
  - estimation error
- Fisher information
- Quantum Fisher information

Necessity of QET -- What is Measurement Error? --

- Quantum measurement

Measurement operators (Kraus operators)

$$\boldsymbol{M} = \{\hat{M}_{i,j}\}_{i,j} \qquad \sum_{i,j} \hat{M}^{\dagger}_{i,j}\hat{M}_{i,j} = \hat{I}$$ i : measurement outcome

$$p(i) := \sum_{j} \mathrm{Tr}[\hat{M}_{ij}\hat{\rho}\hat{M}^{\dagger}_{ij}]$$: probability distribution of measurement outcome

$$\hat{\rho}' = \sum_{i,j} \hat{M}_{ij}\hat{\rho}\hat{M}^{\dagger}_{ij}$$: post-measurement state

- Measurement error = variance of measurement outcomes ???

If $\boldsymbol{M} = \{\hat{M}_{0,j}\}_j$, measurement outcome is always 0.

→ variance of measurement outcome is 0.

However, no information about system is obtained!

In general,

error in measurement ≠ variance of measurement outcome

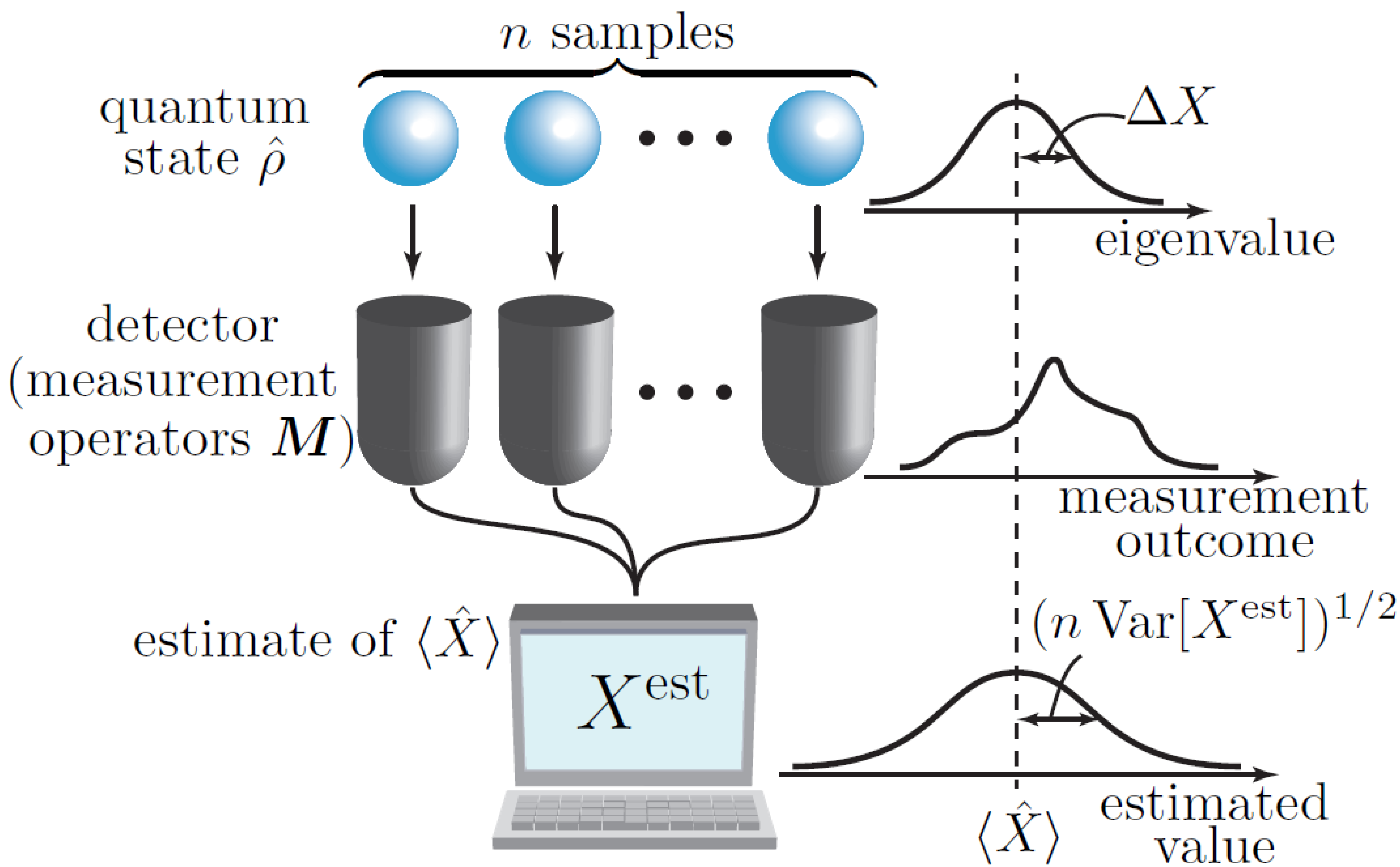
To quantify the error, it is necessary to consider

information obtained by the measurement.

How obtain information?

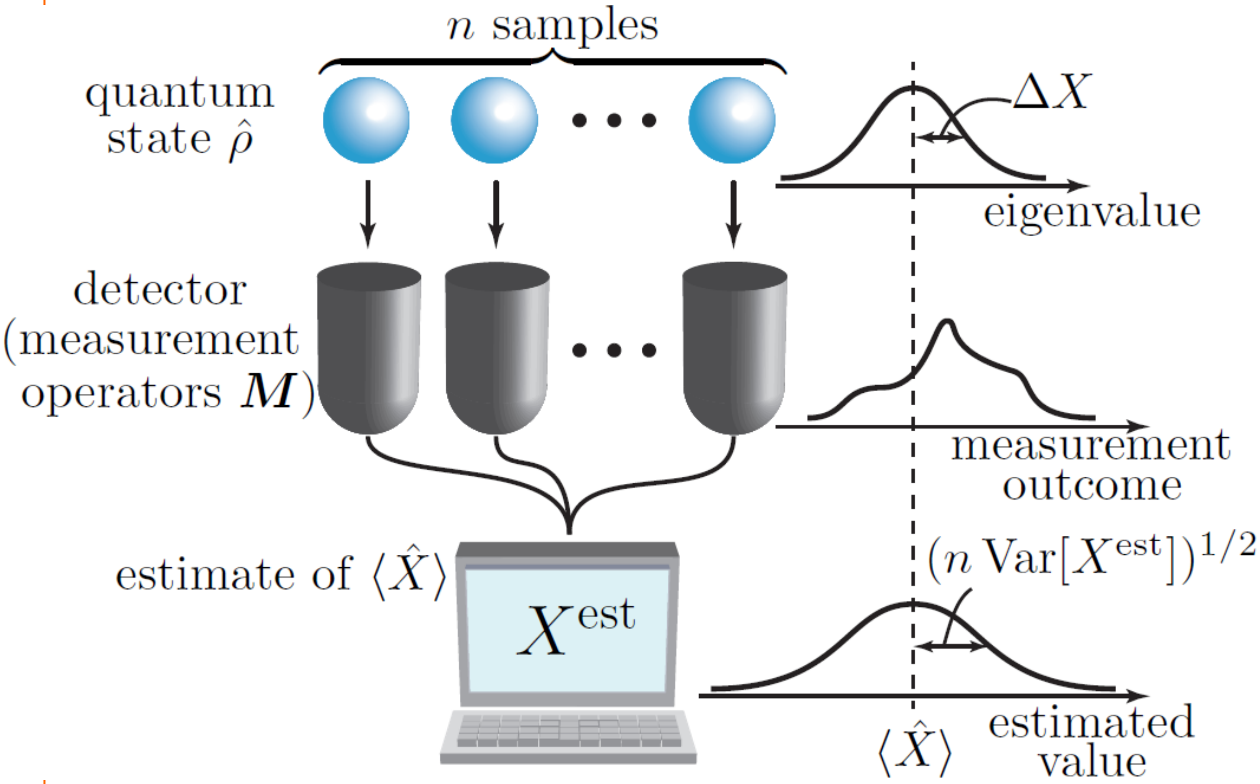
- How to extract information from the measurement outcomes?

⇨ Estimate from the measurement outcomes.



How quantify measurement error?

- Three types of error in estimate process



fluctuation of $\hat{X}$

fluctuation of $\hat{X}$
+ measurement error

fluctuation of estimator
= fluctuation of $\hat{X}$
+ measurement error
+ estimate error

How quantify measurement error from fluctuation of estimator?
➡ Fisher Information (upper bound of accuracy of estimate) and quantum Fisher information (fluctuation of observable) can be used.

Fisher information and Cramér–Rao inequality

If the estimated value converges to the true value:

$$\lim_{n\to\infty} \mathrm{Prob}(|X^{\mathrm{est}} - \langle \hat{X} \rangle| < \delta) = 1 \qquad \text{for all } \hat{\rho} \text{ and } \delta > 0$$

the following inequality is satisfied:

$$\lim_{n\to\infty} n \mathrm{Var}[X^{\mathrm{est}}] \geq \boldsymbol{x} \cdot J(\boldsymbol{M})^{-1} \boldsymbol{x}$$: upper bound of estimate accuracy.

$J(\boldsymbol{M})$: Fisher information matrix

$$J(\boldsymbol{M})_{ij} := \sum_k \frac{\partial \log p(k;\boldsymbol{\theta})}{\partial \theta_i} \frac{\partial \log p(k;\boldsymbol{\theta})}{\partial \theta_j} p(k;\boldsymbol{\theta})$$

: Metric on the space of probability distribution.
Distinguishability of two probability distribution.

$$p(k;\boldsymbol{\theta}) = \sum_l \mathrm{Tr} \hat{\rho} \hat{M}_{k,l}^{\dagger} \hat{M}_{k,l} \qquad x_i := \frac{\partial \langle \hat{X} \rangle}{\partial \theta_i}$$: e.g. direction of spin for d=1/2.

$$\boldsymbol{\theta} = (\theta_1, \ldots, \theta_{d^2-1}) \in \mathbb{R}^{d^2-1} \qquad d = \dim \mathcal{H}$$

: parameter that characterizes $\hat{\rho}$. e.g. Bloch vector for qubit.

Equality can be achieved (e.g. maximum likelihood estimator)

$$\min_{X^{\mathrm{est}}} \lim_{n\to\infty} n \mathrm{Var}[X^{\mathrm{est}}] = \boldsymbol{x} \cdot J(\boldsymbol{M})^{-1} \boldsymbol{x}$$ = fluctuation of $\hat{X}$ + measurement error

Quantum Fisher information and Quantum Cramér–Rao Inequality

- For all quantum measurement,

$$J_Q \geq J(\boldsymbol{M})$$

Upper bound of the obtained information is given.

J_Q : Quantum Fisher information matrix

$\boldsymbol{x} \cdot J_Q^{-1} \boldsymbol{x} = (\Delta \hat{X})^2$ Quantum Fisher information ⟺ fluctuation of $\hat{X}$

fluctuation of $\hat{X}$
\+ measurement error $= \boldsymbol{x} \cdot J(\boldsymbol{M})^{-1} \boldsymbol{x} \geq \boldsymbol{x} \cdot J_Q^{-1} \boldsymbol{x} =$ fluctuation of $\hat{X}$

achieved by the projection measurement of $\hat{X}$

- Definition of measurement error

$$\varepsilon(\hat{X}; \boldsymbol{M}) := \min_{X^{\mathrm{est}}} \lim_{n \to \infty} n \mathrm{Var}[X^{\mathrm{est}}] - (\Delta X)^2$$

$$= \boldsymbol{x} \cdot [J(\boldsymbol{M})^{-1} - J_Q^{-1}] \boldsymbol{x} \geq 0$$

achieved by the projection measurement of $\hat{X}$

We successfully quantify the error in measurement in terms of Fisher information.

Information-theoretic Formulation of Disturbance by using QET

What is Disturbance?

- Disturbance $\neq$ fluctuation of observable on post-measurement state
  - Totally destroyed case: $\boldsymbol{M} = \{|\psi\rangle\langle\varphi_i|\}_i$

    ➡ Post-measurement state is always $|\psi\rangle$

    If $|\psi\rangle$ is a eigenstate of $\hat{Y}$, then fluctuation of $\hat{Y}$ vanishes.

    Howerver, information about original state is totally disappear!

  - Recoverable case: $\boldsymbol{M} = \{\hat{U}\}$

    ➡ Post-measurement state $\hat{U}\hat{\rho}\hat{U}^\dagger$ is recoverable.

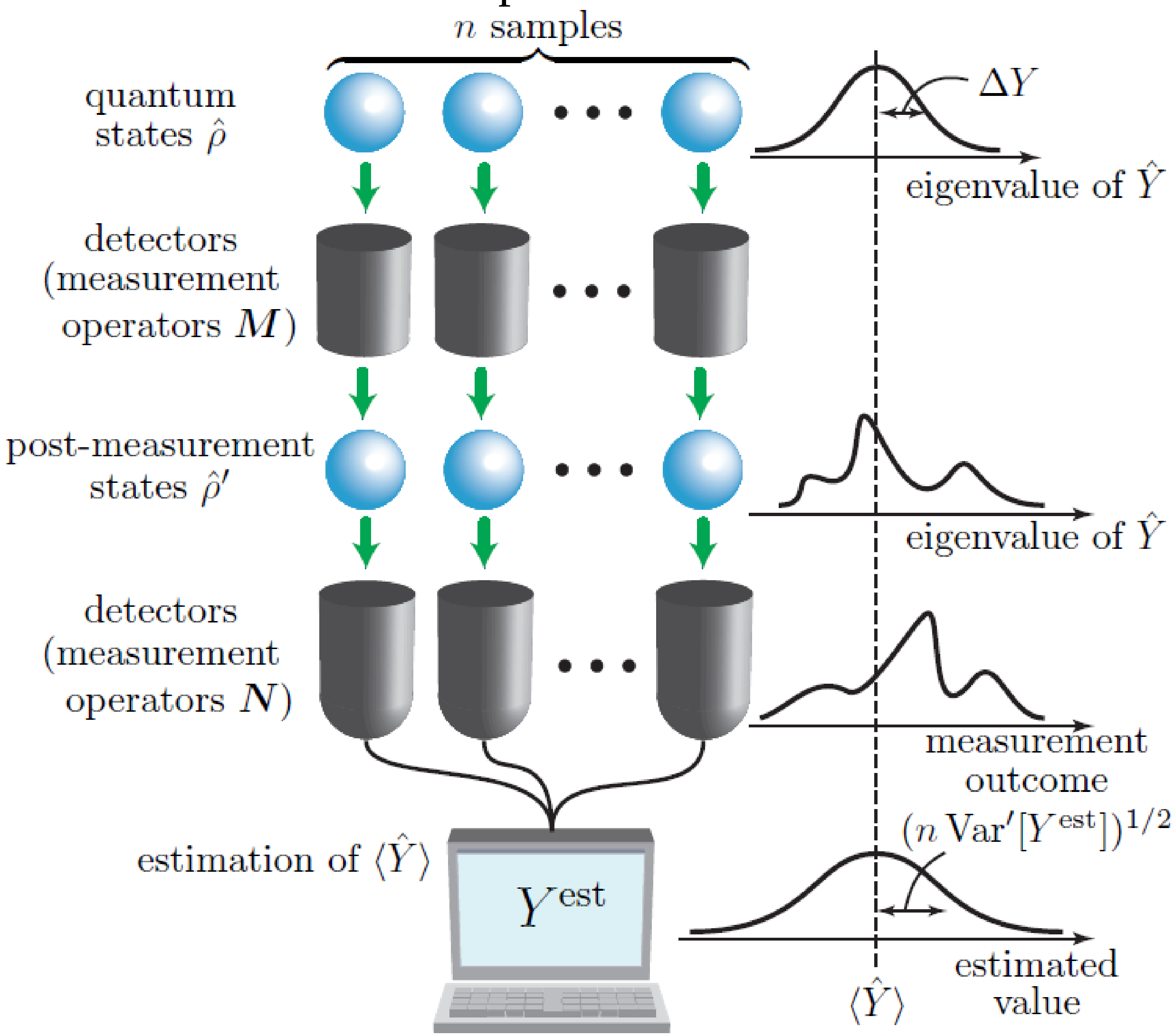
    If $\hat{\rho}$ is a eigenstate of $\hat{Y}$, then fluctuation of $\hat{Y}$ increases.

    However, information about original state is conserved.

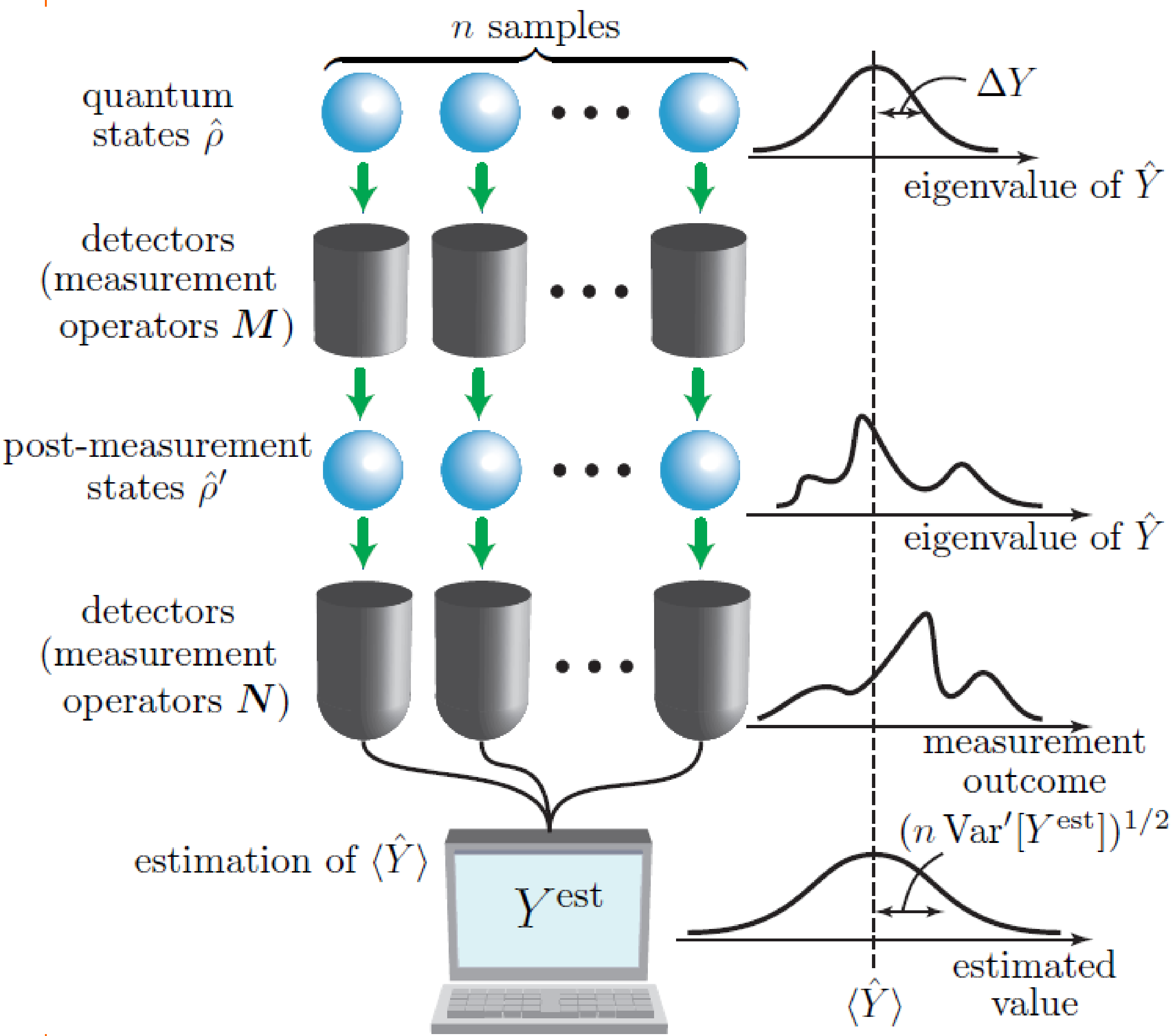
For quantifying disturbance, it is necessary to consider information stored on post-measurement state.

How quantify information on post-measurement?

Consider the second measurement $\boldsymbol{N}$ and estimate process from the measurement outcome.



How quantify disturbance?



fluctuation of $\hat{Y}$

fluctuation of $\hat{Y}$
+ disturbance

fluctuation of $\hat{Y}$
+ disturbance
+ measurement error

fluctuation of estimator
= fluctuation of $\hat{Y}$
+ disturbance
+ measurement error
+ estimate error

Definition of disturbance

eliminate measurement error

eliminate estimate error

$$\eta(\hat{Y};\boldsymbol{M}) := \min_{\boldsymbol{N}} \min_{Y^{\mathrm{est}}} n\mathrm{Var}[Y^{\mathrm{est}}] - (\Delta Y)^2$$

$$= \min_{\boldsymbol{N}} \boldsymbol{y}\cdot[J'(\boldsymbol{N})^{-1} - J_Q^{-1}]\boldsymbol{y}$$

$$= \boldsymbol{y}\cdot[J_Q'^{-1} - J_Q^{-1}]\boldsymbol{y} \geq 0$$

ex. $\boldsymbol{M}$ is projective measurement of $\hat{Y}$

$\boldsymbol{N}$: 2nd measurement

$J'(\boldsymbol{N})$: Fisher information obtained by the 2nd measurement

J_Q' : Quantum Fisher information of post-measurement state

We successfully quantify disturbance by information loss in terms of the Fisher information

Uncertainty Relations between Error and Disturbance

Theorem 1. --- Uncertainty Relation between Error and Disturbance ---

- For all measurements M, observables $\hat{X}, \hat{Y}$, and quantum states $\hat{\rho}$,

$$\varepsilon(\hat{X}; \boldsymbol{M})\eta(\hat{Y}; \boldsymbol{M}) \geq \frac{1}{4}|\langle[\hat{X}, \hat{Y}]\rangle|^2$$

  is satisfied.

  - Formulation of Heisenberg's uncertainty relation by using QET.
  - Error and disturbance is bounded by the commutation relation.

- However, the equality cannot be attained.

  Ex. if $\hat{\rho} = \frac{1}{d}\hat{I}$, $(d = \dim \mathcal{H})$

  $\langle[\hat{X}, \hat{Y}]\rangle = 0$ for all $\hat{X}, \hat{Y}$

What is the attainable bound?

Quantum Fluctuation and Correlation

Extracting inherent fluctuation and correlation in quantum system

$\mathcal{H}_a$: Simultaneous irreducible invariant subspace of $\hat{X}, \hat{Y}$ $\quad \bigoplus_a \mathcal{H}_a = \mathcal{H}$

$\hat{P}_a$: projection on $\mathcal{H}_a$ $\quad \hat{\rho}_a := \frac{1}{p_a}\hat{P}_a\hat{\rho}\hat{P}_a \quad p_a := \mathrm{Tr}[\hat{\rho}\hat{P}_a]$

$$\hat{X} = \left(\begin{array}{c|c} \hat{X}_1 & 0 \\ \hline 0 & \hat{X}_2 \end{array}\right) \qquad \hat{Y} = \left(\begin{array}{c|c} \hat{Y}_1 & 0 \\ \hline 0 & \hat{Y}_2 \end{array}\right) \qquad \hat{\rho} = \left(\begin{array}{c|c} p_1\hat{\rho}_1 & * \\ \hline * & p_2\hat{\rho}_2 \end{array}\right)$$

$$(\Delta_Q X)^2 := \sum_a p_a\{\mathrm{Tr}[\hat{\rho}_a\hat{X}^2] - \mathrm{Tr}[\hat{\rho}_a\hat{X}]^2\}$$: Quantum fluctuation

$$C_Q(\hat{X},\hat{Y}) := \sum_a p_a\{\tfrac{1}{2}\mathrm{Tr}[\hat{\rho}_a\{\hat{X},\hat{Y}\}] - \mathrm{Tr}[\hat{\rho}_a\hat{X}]\mathrm{Tr}[\hat{\rho}_a\hat{Y}]\}$$: Quantum correlation

Example,

- $\hat{X}, \hat{Y}$ cannot be block-diagonalized.（only quantum fluctuation）

  ⇒ $\Delta_Q X = \Delta X \qquad C_Q(\hat{X},\hat{Y}) = C(\hat{X},\hat{Y})$

- $\hat{X}, \hat{Y}$ are commuting each other.

  ⇒ $\Delta_Q X = 0 \qquad C_Q(\hat{X},\hat{Y}) = 0$

Theorem 2. --- Attainable Bound of Error and Disturbance ---

- For all
  - probabilistic projective measurements $\boldsymbol{E} = q\boldsymbol{P}_1 + (1-q)\boldsymbol{P}_2 \qquad 0 \le q \le 1$
  - observables $\hat{X}, \hat{Y}$
  - quantum states $\hat{\rho}$,

$$\varepsilon(\hat{X};\boldsymbol{M})\eta(\hat{Y};\boldsymbol{M}) \ge (\Delta_Q X)^2(\Delta_Q Y)^2 - C_Q(\hat{X},\hat{Y})^2$$

is satisfied, and equality is attainable.

- This bound is stronger than the bound given by commutation relation
  cf. Schrödinger's inequality
  $$(\Delta_Q X)^2(\Delta_Q Y)^2 - C_Q(\hat{X},\hat{Y})^2 \ge \frac{1}{4}|\langle[\hat{X},\hat{Y}]\rangle|^2$$

- The bound is attainable. ⇒ This bound is the strongest.

Conjecture

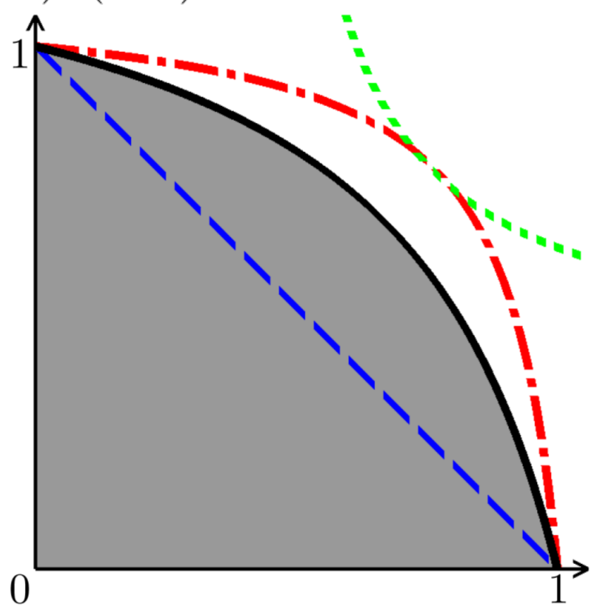
- For all measurement, the following inequality is satisfied.

$$\varepsilon(\hat{X};\boldsymbol{M})\eta(\hat{Y};\boldsymbol{M}) \geq (\Delta_Q X)^2(\Delta_Q Y)^2 - C_Q(\hat{X},\hat{Y})^2$$

- Already proved for $\dim\mathcal{H}=2$.
- For $3\leq\dim\mathcal{H}\leq 7$, we have numerically check the validity.

Error and disturbance of 10^9 randomly generated measurement is plotted.

3,5,7 : prime number
4 : power of prime
6 : composite number



$\dim\mathcal{H}=4\ (S=3/2)\quad \hat{\rho}=\frac{1}{2}\left(\frac{1}{4}\hat{I}+\left|\frac{3}{2}\right\rangle\left\langle\frac{3}{2}\right|\right)$

$\hat{X}=\hat{S}_x \qquad \hat{Y}=\frac{\sqrt{3}}{2}\hat{S}_x+\frac{1}{2}\hat{S}_y$

- Nagaoka's inequality
- $\varepsilon(\hat{X};\boldsymbol{M})\eta(\hat{Y};\boldsymbol{M})=\frac{1}{4}|\langle[\hat{X},\hat{Y}]\rangle|^2$
- $\varepsilon(\hat{X};\boldsymbol{M})\eta(\hat{Y};\boldsymbol{M})=(\Delta_Q X)^2(\Delta_Q Y)^2$
- $\varepsilon(\hat{X};\boldsymbol{M})\eta(\hat{Y};\boldsymbol{M})=(\Delta_Q X)^2(\Delta_Q Y)^2-C_Q(\hat{X},\hat{Y})^2$

Summary

- QET is necessary for formulating error and disturbance.
- We formulate error and disturbance in terms of Fisher information.
- We obtain uncertainty relation between error and disturbance that equality is attainable.

Future issues (after PhD defence…)

- Prove the conjecture.
  (I already prove it partially. Maybe one more step!)
- Find applications and collaborate with experimenters.

Thank you.