Duality and Scales in Quantum-Theoretical Sciences
in Kyoto university

Husimi function
and
Weak measurement

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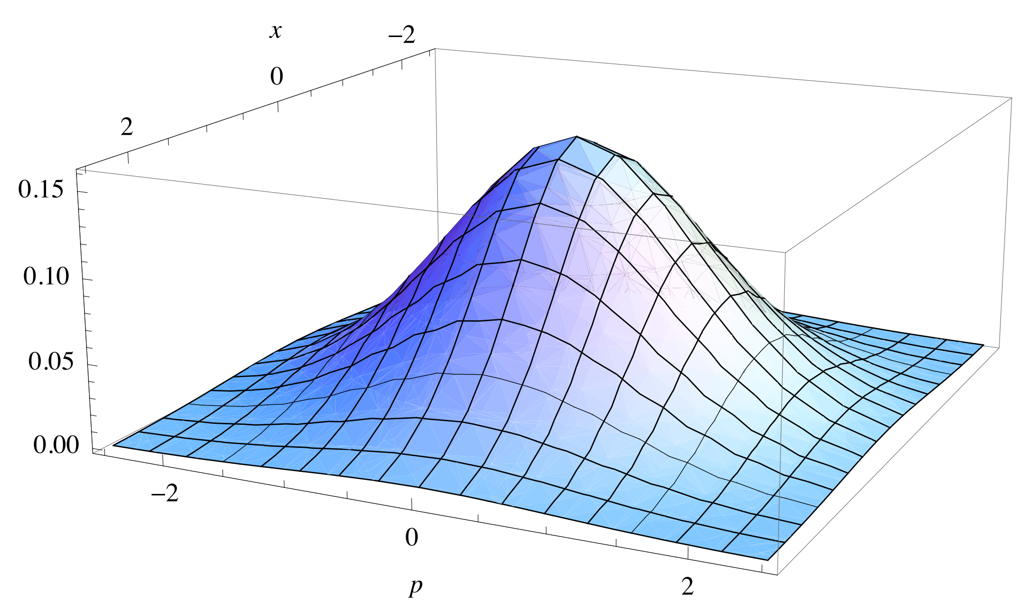
Husimi function $Q(x, p)$

Husimi function is (quasi)distribution function in quantum system

Definition $Q(x,p) = |\langle x,p|\psi\rangle|^2$

$|x,p\rangle$ is coherent state, $|\psi\rangle$ is state



Husimi function of vacuum state

Useful relations of Husimi function

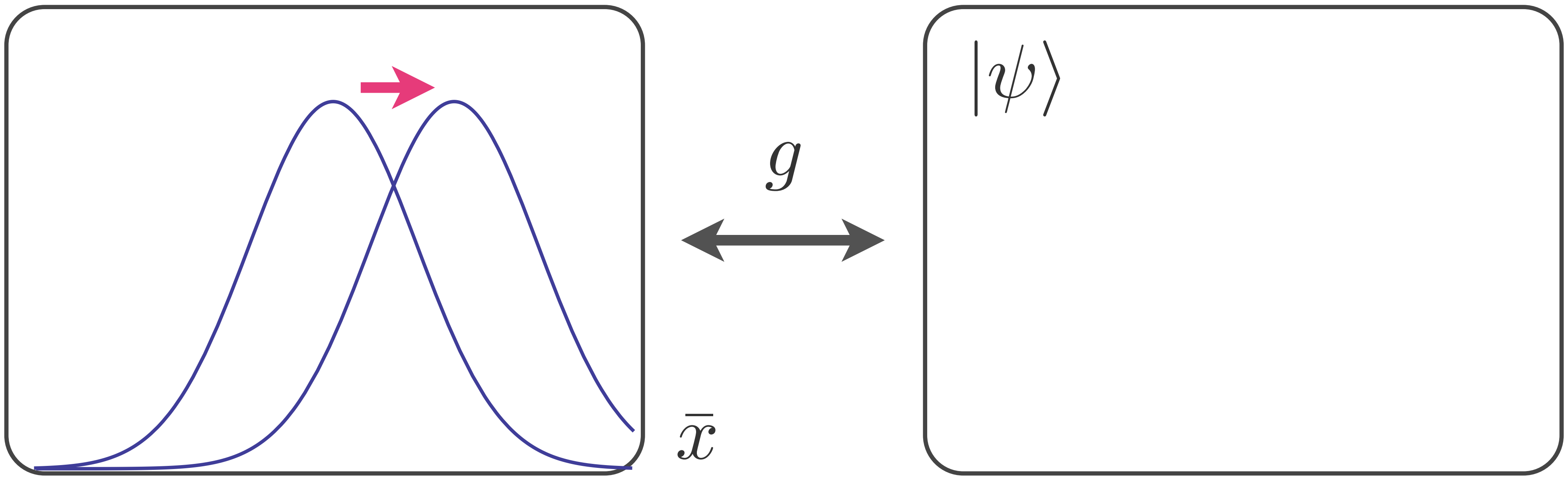
$$\int \frac{dxdp}{2\pi} Q(x,p) = 1 \qquad Q(x,p) \geq 0$$

$$\begin{aligned} Q(x,p) &= \frac{1}{\sqrt{\pi}} \int dx' dx'' \langle \psi | x' \rangle \langle x'' | \psi \rangle \\ &\times \exp\left\{ -\frac{1}{2}(x-x')^2 - \frac{1}{2}(x-x'')^2 + ip(x'-x'') \right\} \end{aligned}$$

Marginal distribution

$$\int \frac{dp}{2\pi} Q(x,p) = \int \frac{dx'}{\sqrt{\pi}} | \langle x' | \psi \rangle |^2 \exp\{-(x-x')^2\}$$

Weak measurement without post selection



$f(\bar{x})$ **gaussian**

a is eigenvalue of $\hat{A}$

$$|\Psi\rangle = \int d\bar{x}\, f(\bar{x})\, |\bar{x}\rangle\, |\psi\rangle$$

$$\hat{U} = e^{-ig\hat{\bar{p}}\hat{A}}$$

$$|\Psi'\rangle = \sum_a \int d\bar{x}\, \langle a|\psi\rangle\, f(\bar{x} - ga)\, |\bar{x}\rangle\, |a\rangle$$

$$\delta(\bar{x} - \hat{x}) \otimes \hat{I}$$

$$\langle\Psi'|\,(\delta(\bar{x} - \hat{x}) \otimes \hat{I})\,|\Psi'\rangle = \sum_a |\langle a|\psi\rangle|^2 f^2(\bar{x} - ga)$$

$$\simeq f^2(\bar{x} - g\,\langle\hat{A}\rangle) \qquad (g\,\langle\hat{A}\rangle \ll 1)$$

Getting the marginal distribution of Husimi function by weak measurement of position

$$|\Psi\rangle = \frac{1}{\pi^{1/4}} \int d\bar{x} \exp\left\{ -\frac{1}{2}\bar{x}^2 \right\} |\bar{x}\rangle \, |\psi\rangle$$

$$\hat{U} = e^{-i\hat{\bar{p}}\hat{x}}$$

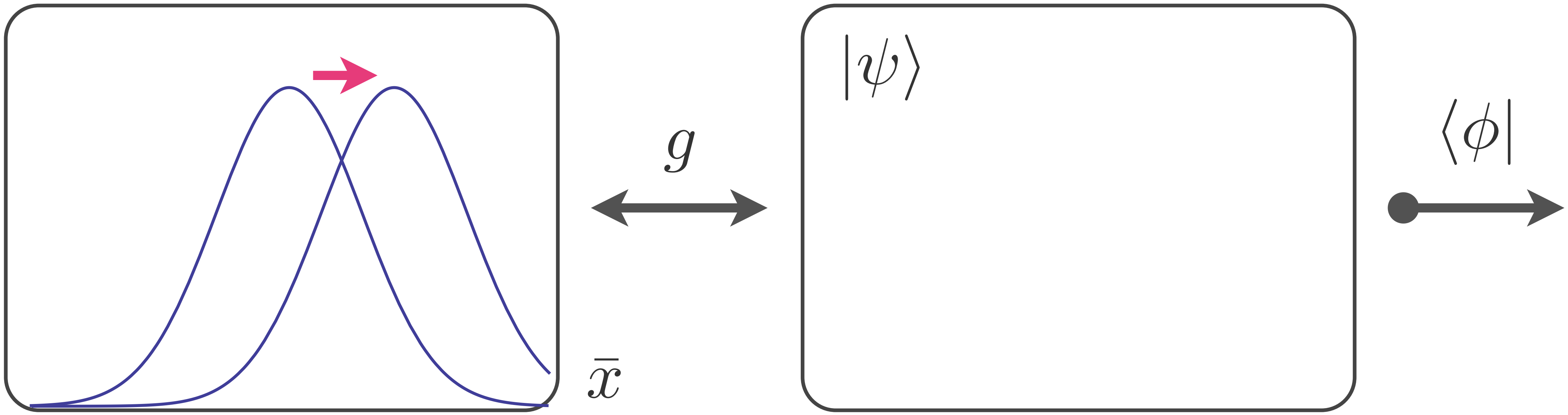
$$\delta(\bar{x} - \hat{x}) \otimes \hat{I}$$

$$|\Psi'\rangle = \frac{1}{\pi^{1/4}} \int dx d\bar{x} \, \langle x|\psi\rangle \exp\left\{ -\frac{1}{2}(\bar{x} - x)^2 \right\} |\bar{x}\rangle \, |x\rangle$$

$$\langle\Psi'| \left(\delta(\bar{x} - \hat{x}) \otimes \hat{I}\right) |\Psi'\rangle = \int \frac{dx}{\sqrt{\pi}} |\langle x|\psi\rangle|^2 \exp\{-(\bar{x} - x)^2\}$$

$$\int \frac{dp}{\pi} Q(x,p) = \int \frac{dx'}{\sqrt{\pi}} |\langle x'|\psi\rangle|^2 \exp\{-(x - x')^2\}$$

Weak measurement with post selection



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$$\langle\phi| \langle\bar{x}|$$

$$(\langle\psi| \langle\bar{x}|) |\Psi'\rangle = \sum_a \langle\phi|a\rangle \langle a|\psi\rangle f(\bar{x} - ga) \simeq f(\bar{x} - g\hat{A}_w)$$

$$\hat{A}_w = \frac{\langle\phi| \hat{A} |\psi\rangle}{\langle\phi|\psi\rangle} \qquad \hat{A}_w \ll 1$$

Weak measurement of position and measurement of momentum, what will we get?

$$|\Psi\rangle = \frac{1}{\pi^{1/4}} \int d\bar{x} \exp\left\{ -\frac{1}{2}\bar{x}^2 \right\} |\bar{x}\rangle |\psi\rangle$$

$$\hat{U} = e^{-i\hat{\bar{p}}\hat{x}}$$

$$|\Psi'\rangle = \frac{1}{\pi^{1/4}} \int dx d\bar{x} \, \langle x|\psi\rangle \exp\left\{ -\frac{1}{2}(\bar{x}-x)^2 \right\} |\bar{x}\rangle |x\rangle$$

$$\langle p| \langle \bar{x}|$$

$$\begin{aligned} f(\bar{x}, p) &= (\langle p| \langle \bar{x}|) |\Psi'\rangle \\ &= \frac{1}{\pi^{1/4}} \int dx \, \langle x|\psi\rangle \exp\left\{ -\frac{1}{2}(\bar{x}-x)^2 - ipx \right\} \end{aligned}$$

We get Husimi function

$$
\begin{aligned}
|f(\bar{x},p)|^2 &= \frac{1}{\sqrt{\pi}} \int dx' dx'' \, \langle \psi | x' \rangle \, \langle x'' | \psi \rangle \\
&\times \exp\left\{ -\frac{1}{2}(\bar{x} - x')^2 - \frac{1}{2}(\bar{x} - x'')^2 + ip(x' - x'') \right\}
\end{aligned}
$$

$$
\begin{aligned}
Q(x,p) &= \frac{1}{\sqrt{\pi}} \int dx' dx'' \, \langle \psi | x' \rangle \, \langle x'' | \psi \rangle \\
&\times \exp\left\{ -\frac{1}{2}(x - x')^2 - \frac{1}{2}(x - x'')^2 + ip(x' - x'') \right\}
\end{aligned}
$$

Husimi function by weak measurement is one of Naimark's dilation, but differ from conventional discussion.

My discussion

$$\delta(\hat{p}_1 - p)\delta(\hat{x}_2 - x) \qquad \hat{U}\,|\psi_1\rangle\,|\psi_2\rangle$$

measurement system prove

Conventional discussion on Husimi function and Naimark's dilation

$$\delta(\hat{x}_1 - \hat{x}_2 - x)\delta(\hat{p}_1 + \hat{p}_2 - p) \qquad |\psi_1\rangle\,|\psi_2\rangle$$

measurement system prove

Summary

1. Husimi function is given by weak measurement of position and measurement of momentum.

2. Our result is related to ‘Naimark’s dilation’, but differ from Conventional discussion.

3. To discuss relation between Husimi function and Weak measurement with weak values is future problem.