Discrete Time Quantum Walk with Incommensurate Coin

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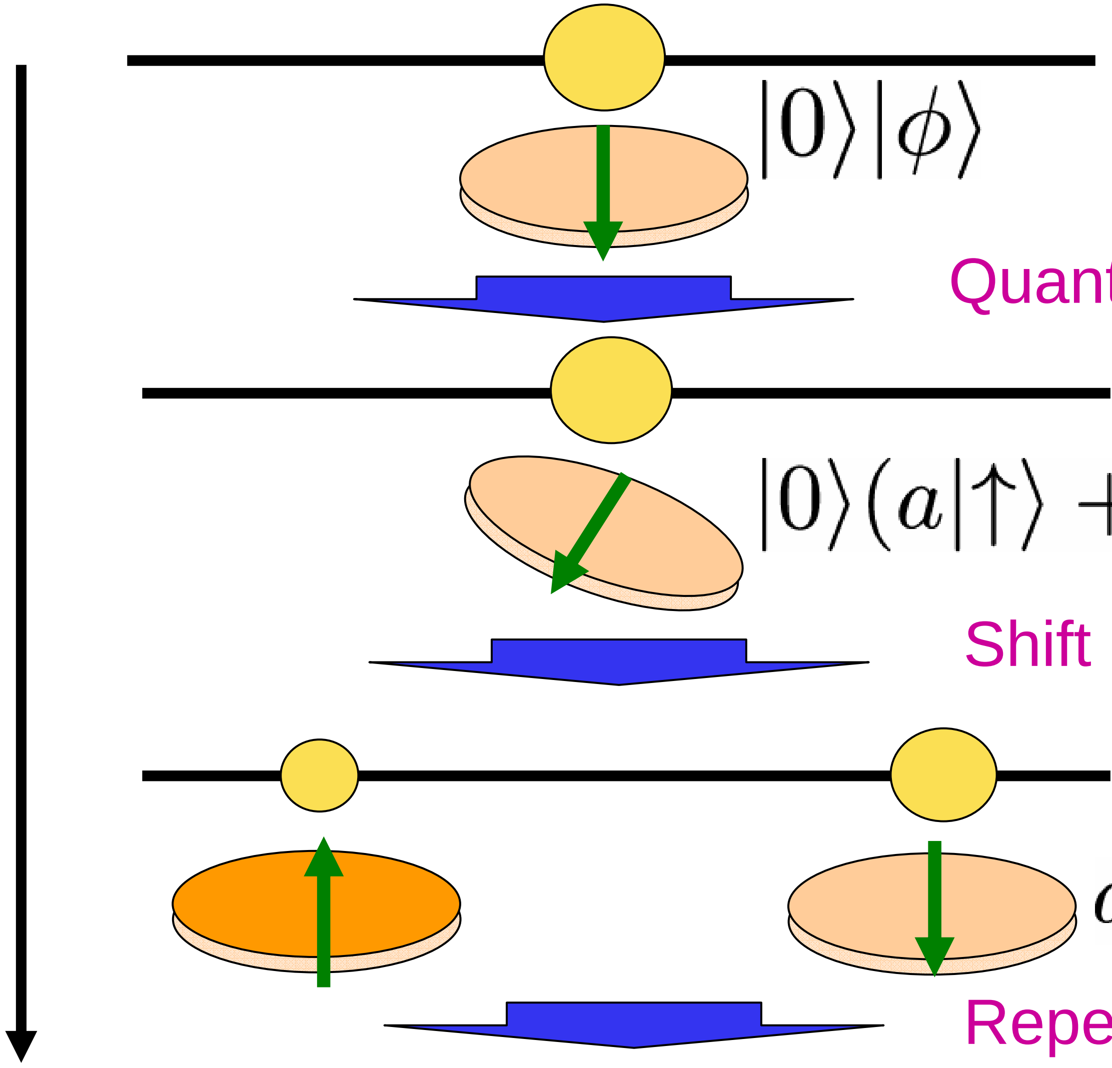
Discrete Time Quantum Walk (DTQW)

(A. Ambainis, E. Bach, A. Nayak, A. Vishwanath, and J. Watrous, in *STOC'01* (ACM Press, New York, 2001), pp. 37 – 49.)



$$|0\rangle|\phi\rangle$$

Quantum Coin Flip

$$C \in U(2)$$

$$|0\rangle(a|\uparrow\rangle + b|\downarrow\rangle)$$

Shift

$$W|n,\uparrow\rangle = |n-1,\uparrow\rangle$$
$$W|n,\downarrow\rangle = |n+1,\downarrow\rangle$$

$$a|-1\rangle|\uparrow\rangle + b|+1\rangle|\downarrow\rangle$$

Repeat $U = W(I \otimes C)$

$$\Pr(n;t) = \sum_{\xi \in \{\uparrow,\downarrow\}} |\langle n,\xi|U^t|0,\phi\rangle|^2$$

Probability Distribution at the 1000-th step

Random Walk

Unbiased Coin

(Left and Right
with probability ½)

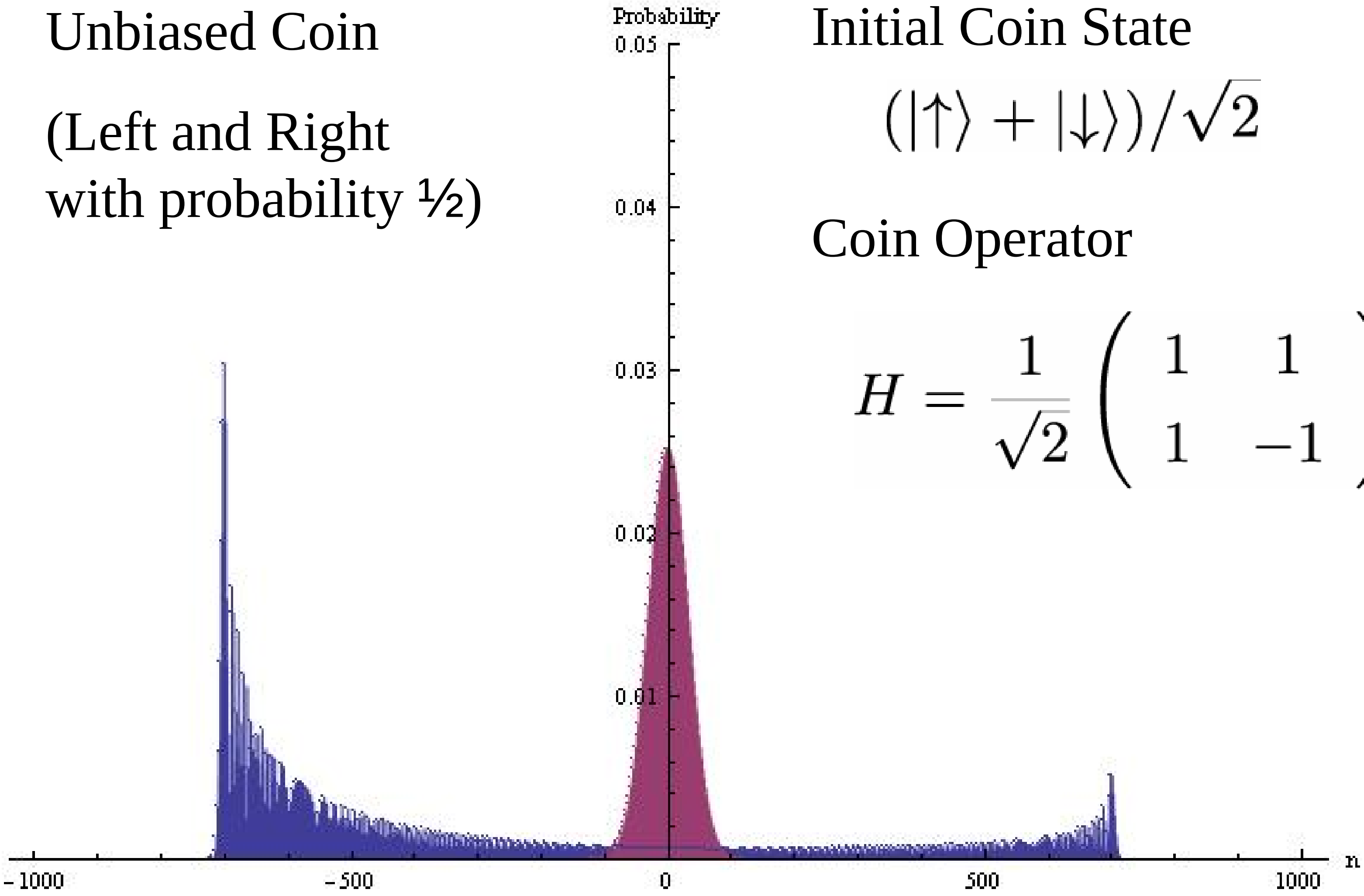
Quantum Walk

Initial Coin State

$$(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$$

Coin Operator

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$



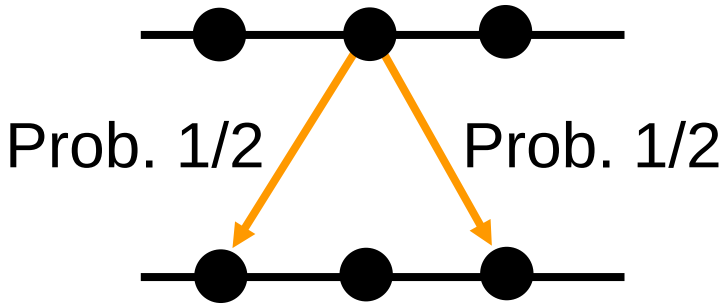
Weak Limit Theorem (Limit Distribution)

(Convergence in terms of the distribution)

Random Walk

Central Limit Theorem

$$\frac{X_t}{\sqrt{t}} \Rightarrow N(0,1)$$



Quantum Walk

N. Konno, Quantum Information Processing **1**, 345 (2002)

$$\frac{X_t}{t} \Rightarrow K(|a|)$$

Coin operator

$$C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Initial state

$$\alpha|\uparrow\rangle + \beta|\downarrow\rangle$$

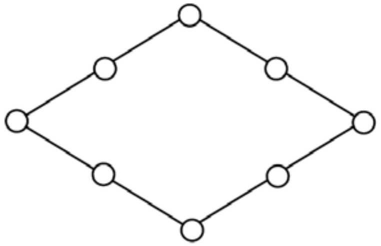
Probability density

$$\Pr(x) = \frac{\sqrt{1-|a|^2}}{\pi(1-x^2)\sqrt{|a|^2-x^2}}\left\{1-\left(|\alpha|^2-|\beta|^2+\frac{a\alpha\overline{b\beta}+\overline{a\alpha}b\beta}{|a|^2}\right)x\right\}$$

for $x \in (-|a|, |a|)$

Experimental and Theoretical Progresses

- Experimental Realizations
  - Trapped Atoms with Optical Lattice and Ion Trap
    - M. Karski *et al.,* Science **325**, 174 (2009). 23 step
    - F. Zahringer *et al.*, Phys. Rev. Lett. **104**, 100503 (2010). 15 step
  - Photon in Linear Optics and Quantum Optics
    - A. Schreiber *et al.,* Phys. Rev. Lett. **104**, 050502 (2010). 5 step
    - M. A. Broome *et al.*, Phys. Rev. Lett. **104**, 153602. 6 step
    - A. Peruzzo *et al.*, Science **329**, 5998 (2010). 10 step (2-particle)
  - Molecule by NMR
    - C. A. Ryan, M. Laforest, J. C. Boileau, and R. Laflamme, Phys. Rev. A **72**, 062317 (2005). 8 step
- Applications
  - Universal Quantum Computation
    - N. B. Lovett *et al.,* Phys. Rev. A **81**, 042330 (2010).



  - Quantum Simulator
    - T. Oka, N. Konno, R. Arita, and H. Aoki, Phys. Rev. Lett. **94**, 100602 (2005). (Landau-Zener Transition)
    - T. Kitagawa, M. Rudner, E. Berg, and E. Demler, Phys. Rev. A **82**, 033429 (2010). (Topological Phase)

Last-Year Question at RIMS/YITP Conference

- How to deal with the DTQW with decoherence?
  - YS, K. Chisaki, E. Segawa, and N. Konno, Phys. Rev. A **81**, 062129 (2010).
    - Decoherence Mechanism and Crossover to Random Walk
  - K. Chisaki, N. Konno, E. Segawa, and YS, arXiv:1009.2131.
    - Connection to Continuous Time Quantum Walk
    - Crossover to Continuous Random Walk
    - Crossover to Lazy Random Walk
      - From the DTQW with the time dependent coin.
  - M. Gonulol, E. Aydiner, YS, O. Mustecaplioglu, arXiv:1008.0085.
    - Multi-particle Quantum Walk on K cycle with trapped sites
    - Quantum Phase Transition-like Behavior in Thermodynamic Limit

Today's Question

- We see the ballistic transport in the DTQW in the homogeneous coin case.
- However, in the inhomogeneous case, what behaviors should we see?

Our Model Self-dual model inspired by the Aubry-Andre model

$$C = \sum_n \left[|n\rangle\langle n| \otimes \begin{pmatrix} \cos(2\pi\alpha n) & -\sin(2\pi\alpha n) \\ \sin(2\pi\alpha n) & \cos(2\pi\alpha n) \end{pmatrix} \right]$$

In the dual basis, the roles of coin and shift are interchanged.

Dual basis

$$|n, \tilde{\uparrow}\rangle = \sum_m [\sin(2\pi\alpha mn)|m, \uparrow\rangle + \cos(2\pi\alpha mn)|m, \downarrow\rangle]$$

$$|n, \tilde{\downarrow}\rangle = \sum_m [\cos(2\pi\alpha mn)|m, \uparrow\rangle + \sin(2\pi\alpha mn)|m, \downarrow\rangle]$$

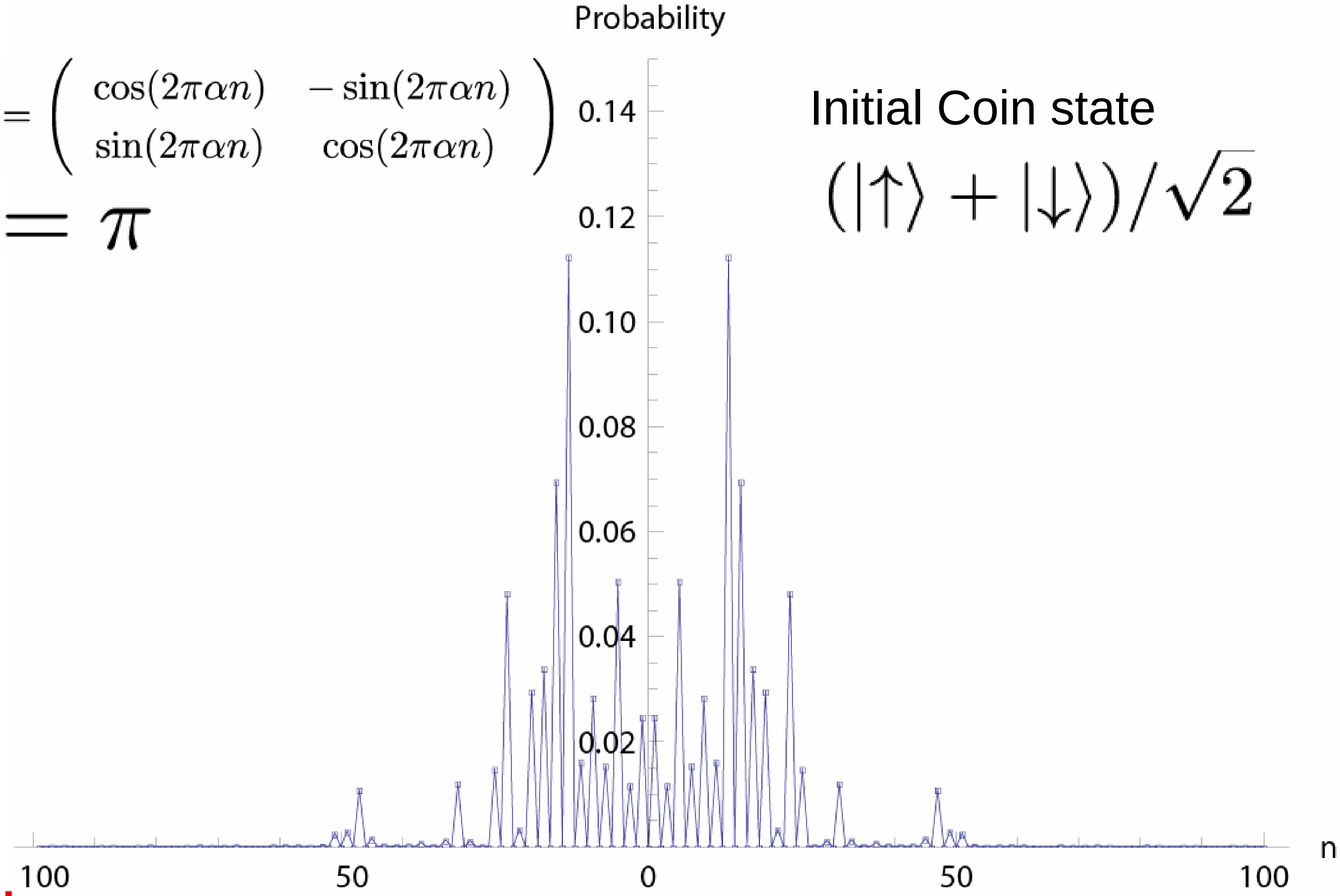
Probability Distribution at the 1000-th Step

$$C(n) = \begin{pmatrix} \cos(2\pi\alpha n) & -\sin(2\pi\alpha n) \\ \sin(2\pi\alpha n) & \cos(2\pi\alpha n) \end{pmatrix}$$

$$\alpha = \pi$$

Initial Coin state

$$(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$$



www.rle.mit.edu

Limit Distribution

Theorem

(YS and H. Katsura, Phys. Rev. E **82**, 031122 (2010))

For any irrational $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and any $\alpha = \frac{P}{4Q} \in \mathbb{Q}$ with relatively prime P (odd integer) and Q, the limit distribution of the inhomogeneous QW divided by any power of the time variable is localized at the origin:

$$\frac{X_t}{t^\theta} \Rightarrow \delta(x) \quad (t \to \infty),$$

where X_t is the random variable for the position at the t step, θ (> 0) is an arbitrary parameter, and $\delta(\cdot)$ is the Dirac delta function. Note that, "$\Rightarrow$" means convergence in distribution.

Proof Methods

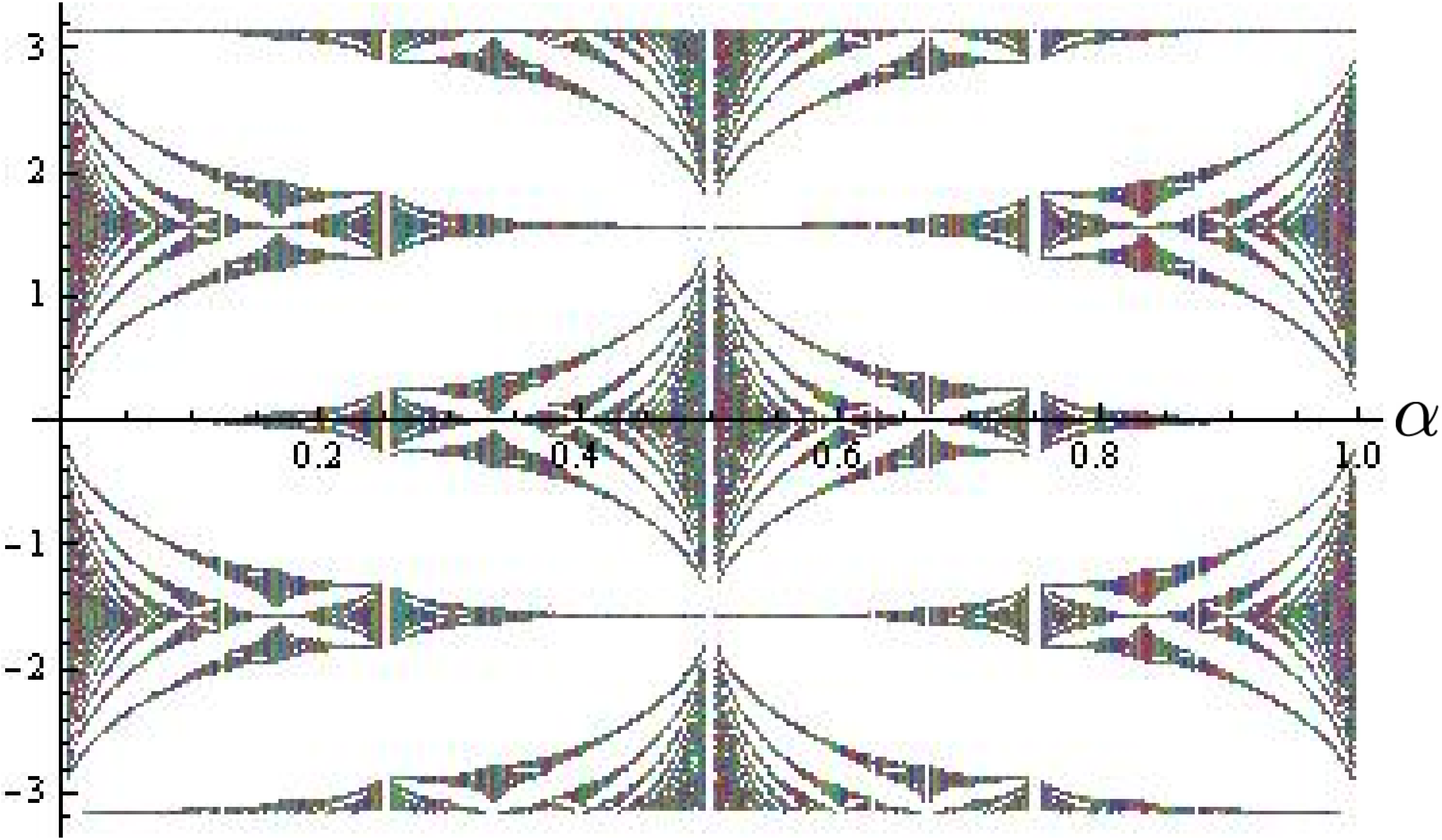
1. The diagonal elements of the coin must be zero for some positions in the rational number case.
2. Dirichlet's Approximation Theorem

- The Aubry-Andre model is related to the Hofstadter butterfly, which has a self-similar and fractal structure.

- The quantum walk operator WC is unitary.
  - The absolute value of the eigenvalues of WC is 1.
- The eigenvalues of WC and CW are same.

- We numerically obtain the distribution of the eigenvalues of CW for the parameter α .

Distribution of Eigenvalues



Argument of CW

Analytical Properties

(YS and H. Katsura, to be published in AIP Conf. Proc. (2011))

1. All the eigenvalues at α are identical to those at $1 - \alpha$.

2. The eigenvalues come in complex conjugate pairs: for every eigenvalue λ, there is an eigenvalue λ^*. Unitarity

3. The eigenvalues come in chiral pairs: for every eigenvalue λ, there is an eigenvalue $-\lambda$. Local Gauge transformation

4. All the eigenvalues are simple, i.e., nondegenerate.

   Weaker condition of Perron-Frobenius Theorem

5. There are four eigenvalues $\lambda = \pm 1, \pm i$ for any $\alpha = \frac{P}{4Q} \in \mathbb{Q}$.

   Existence of Unmoved mode of DTQW

6. Every eigenvalue λ at $\alpha = \frac{P}{4Q} \in \mathbb{Q}$ corresponds to an eigenvalue $i\lambda$ at $\alpha + 1/2$.

De-Localization Criterion

Theorem (Extended RAGE theorem)

The distribution of the DTQW is not localized if the quantum walk operator WC only has continuous spectra and does not have embedded eigenvalues.

On the RAGE (Ruelle-Amrein-Geogerscu-Enss) Theorem.
B. Simon and M. Reed, *Methods of Modern Mathematical Physics Vol.III Scattering Theory*. (Academic Press, 1977).

Conjectured by
YS and H. Katsura, Phys. Rev. E **82**, 031122 (2010).

Proven by
A. Ahlbrecht, V. Scholz, and A. Werner, work in progress.

Open Question: even if ??

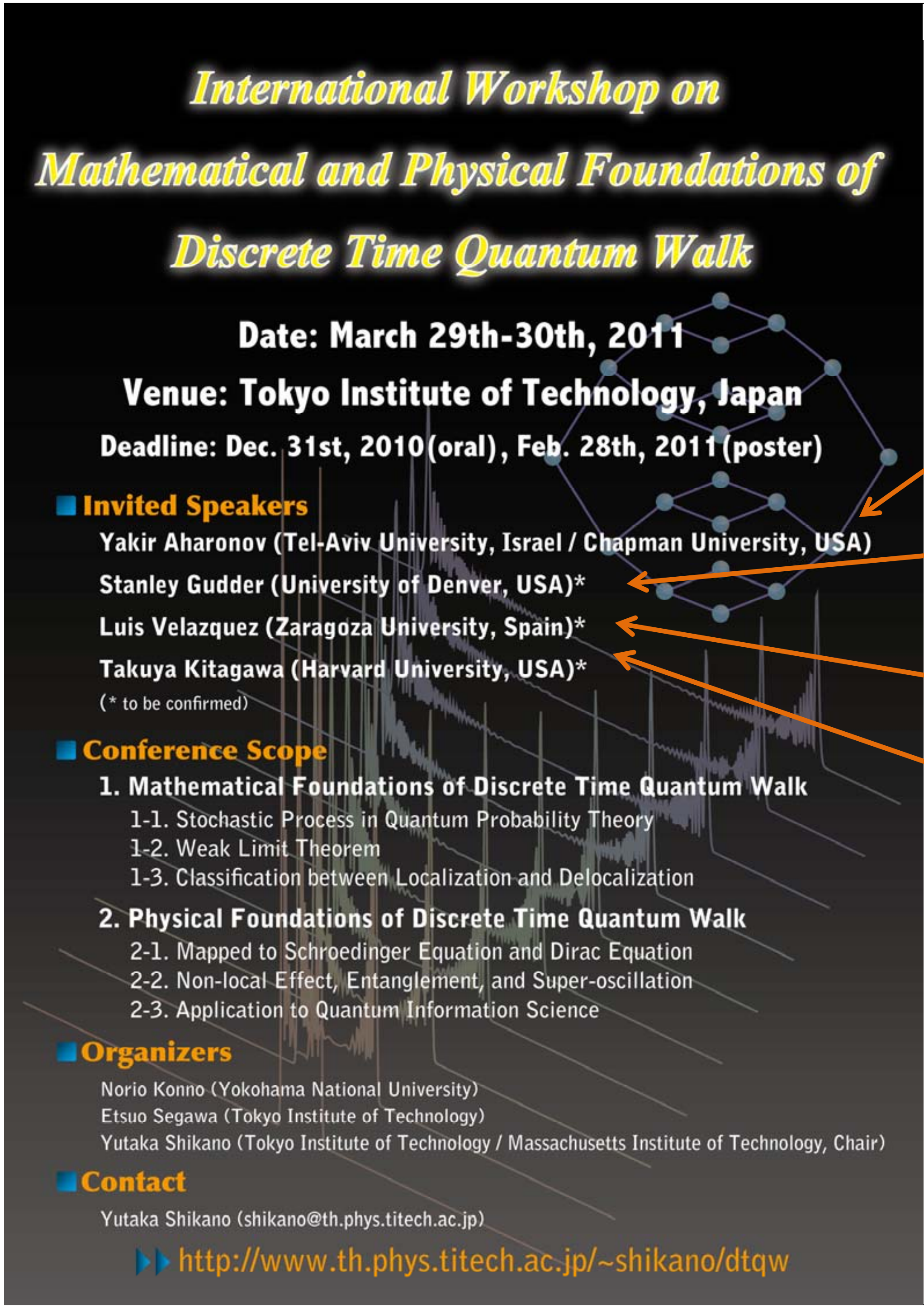
Conclusion

- From the limit distribution,
  - We showed the localization in the incommensurate coin model with self dual.
- From the distribution of the eigenvalues,
  - We showed the fractal and self-similar properties.
  - We proposed on the localization / delocalization in the quantum walk.
- Further Open Questions:
  - Recurrence Properties?
  - Relationship to Random Coin Case?

    (A. Joye and M. Merkli, J. Stat. Phys. **140**, 1025 (2010))
  - Relationship to Time-dependent Coin Case?

    (A. Joye, arXiv:1010.4006)

Please see more details in Phys. Rev. E **82**, 031122 (2010).



http://www.th.phys.titech.ac.jp/~shikano/dtqw/

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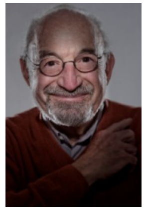

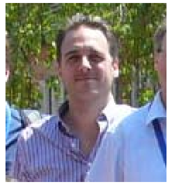

Organizers