Universality classes and the Anderson transition in 2 dimensions

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Nov. 2010 @ Kyoto Univ.

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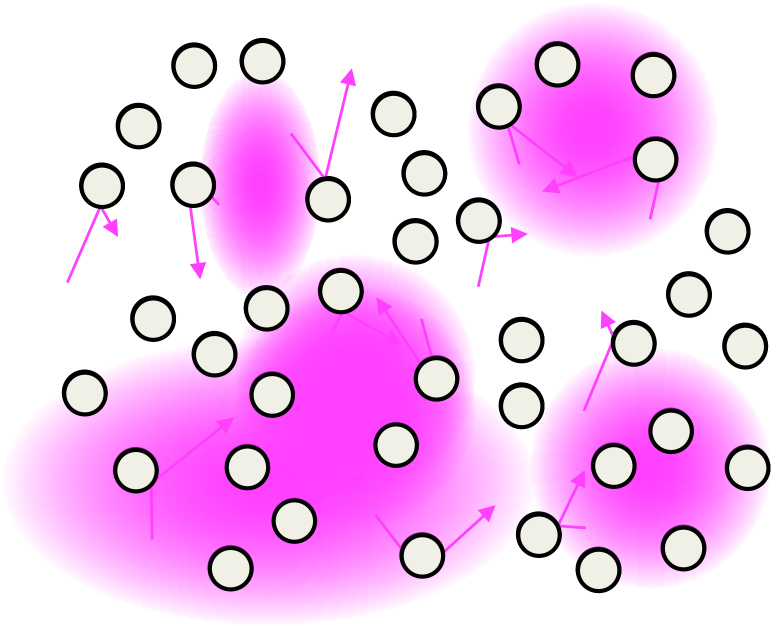
outline

- General theory for the Anderson transition
  - Renewed interest due to BEC experiments
  - Scaling theory
  - Classification according to universality classes
- Anderson transition in 2D (symplectic class)
  - Strange metallic behavior
  - Just at the transition;
    - Conductance distribution, multifractal wave function, conformal invariance
  - Lower critical dimension
  - Topological insulator that carries spin current
- Quantum Hall effect (unitary class)

Anderson localization

- Wave+randomness→constructive interference

→localization of wave (P.W.Anderson, '58)

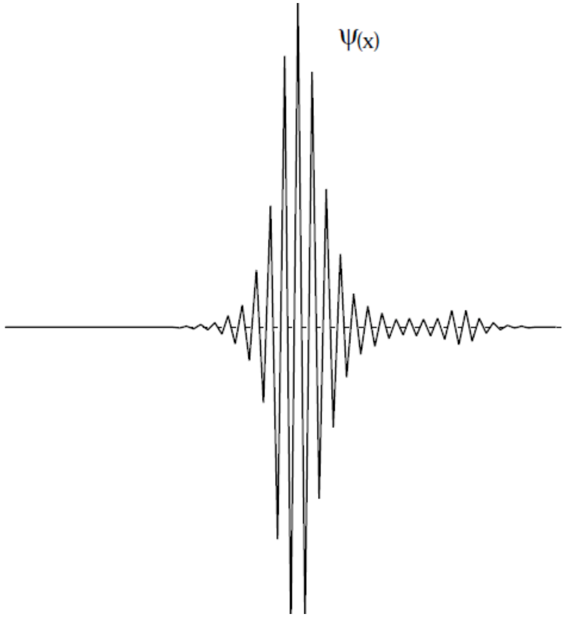


Anderson transition

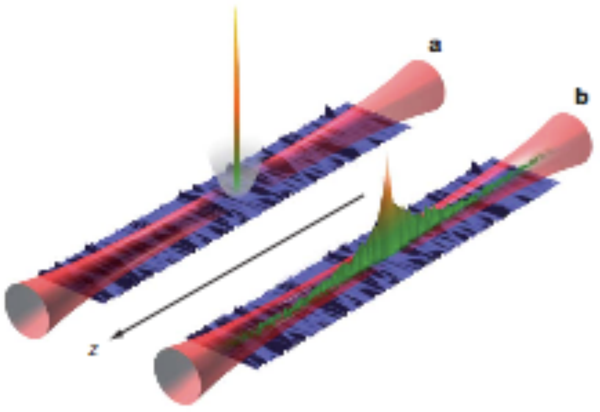
- is a localization-delocalization transition of the eigenfunctions (at the Fermi energy)
- Characterized by the divergence of the localization length

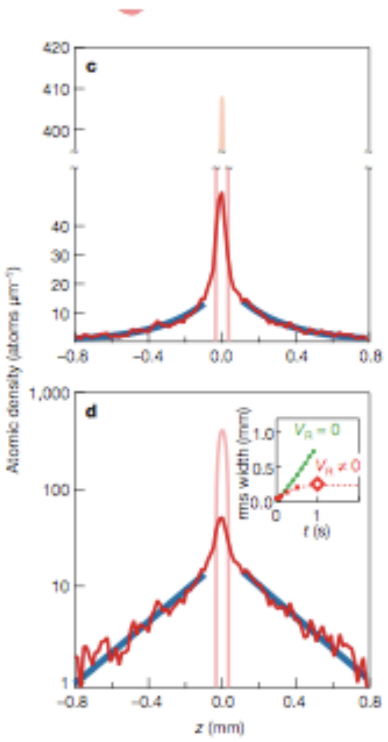
$$\Psi(x) = a(x)\exp\left(-\frac{|x - x_0|}{\xi} \right)$$

$$\xi \sim \frac{1}{|E - E_{\mathrm{c}}|^{\nu}}, \frac{1}{|W - W_{\mathrm{c}}|^{\nu}}$$



Matter wave (real space)



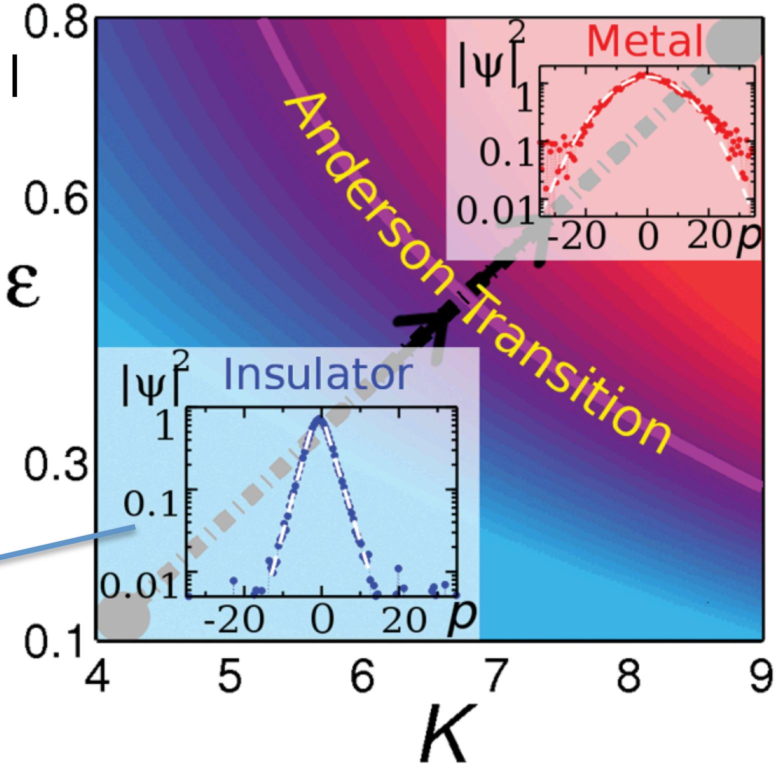


Billy et al., nature Vol 453 (2008) 891
Direct observation of 1d localization (Prof. Kotani, yesterday’s talk)
10^4 Rb atoms

Matter wave (momentum space)

$$H = \frac{p^2}{2} + K \cos x[1 + \varepsilon \cos(\omega_2 t) \cos(\omega_3 t)] \sum_{n=0}^{N-1} \delta(t - n),$$

$$H = \sum_j \varepsilon_j \mid j >< j \mid + \sum_j V_{j',j} \mid j' >< j \mid$$



Mapping kicked rotor to 3D Anderson model in momentum space

10^7 Cs atoms, 3.2 microkelvin

$$\exp(-|p|/p_{\mathrm{loc}})$$

Chabe et al., PRL ‘08

β Function

Nondimensional quantity Q_p

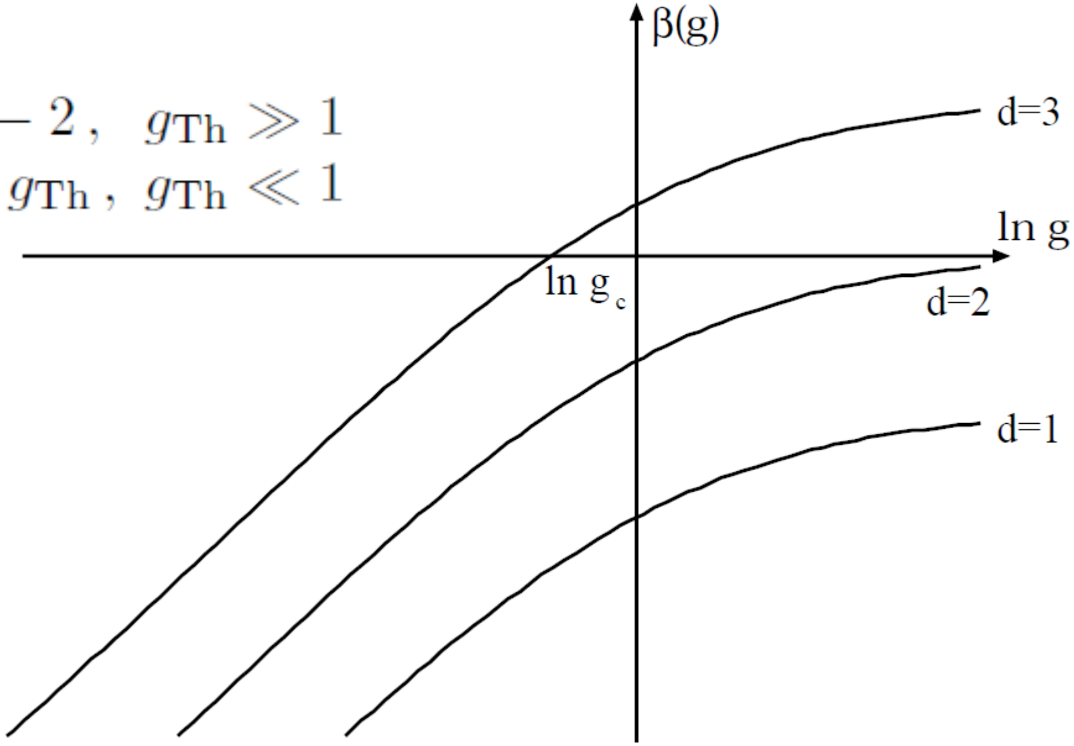
$$\frac{\mathrm{d}Q_{\mathrm{P}}}{\mathrm{d}\ln L} = \beta\left(Q_{\mathrm{P}}\right)$$

Example: log of Thouless conductance

$$g_{\mathrm{Th}} \sim \begin{cases} L^{d-2} \\ \exp(-2L/\xi) \end{cases}$$

$$\beta = \frac{\mathrm{d}\ln g_{\mathrm{Th}}}{\mathrm{d}\ln L}$$

$$\beta = \begin{cases} d-2\,, & g_{\mathrm{Th}} \gg 1 \\ \ln g_{\mathrm{Th}}\,, & g_{\mathrm{Th}} \ll 1 \end{cases}$$



Abrahams et al., PRL ‘79

Classification of the Anderson localization

- Classified to 3 universality classes according to whether the time reversal symmetry (TRS) and spin rotation symmetry (SRS) are preserved or not.（Wigner-Dyson classes: c.f. random matrix theory）
  - Broken TRS due to magnetic field or magnetic impurities→Unitary class
  - TRS but no SRS due to spin-orbit interaction → Symplectic class
  - TRS and SRS→Orthogonal class

Model Hamiltonian

Non-interacting, random Hamiltonian

$$H = \sum_j \varepsilon_j \mid j >< j \mid + \sum_j V_{j',j} \mid j' >< j \mid$$

In magnetic fields

$$V_{jj'} = V \exp\left[\mathrm{i}\frac{e}{\hbar}\int_j^{j'} \mathbf{A}\cdot \mathrm{dx}\right]$$

In the presence of spin-orbit interaction

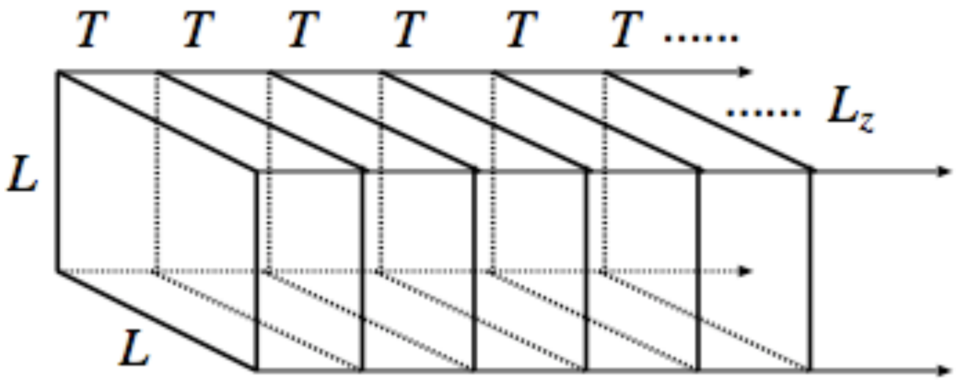
$$H = \sum_{j,\sigma} \varepsilon_j \mid j,\sigma >< j,\sigma \mid + \sum_{j,\sigma,\sigma'} V_{j',\sigma',j,\sigma} \mid j',\sigma' >< j,\sigma \mid$$

Lyapunov exponent

Transfer matrix

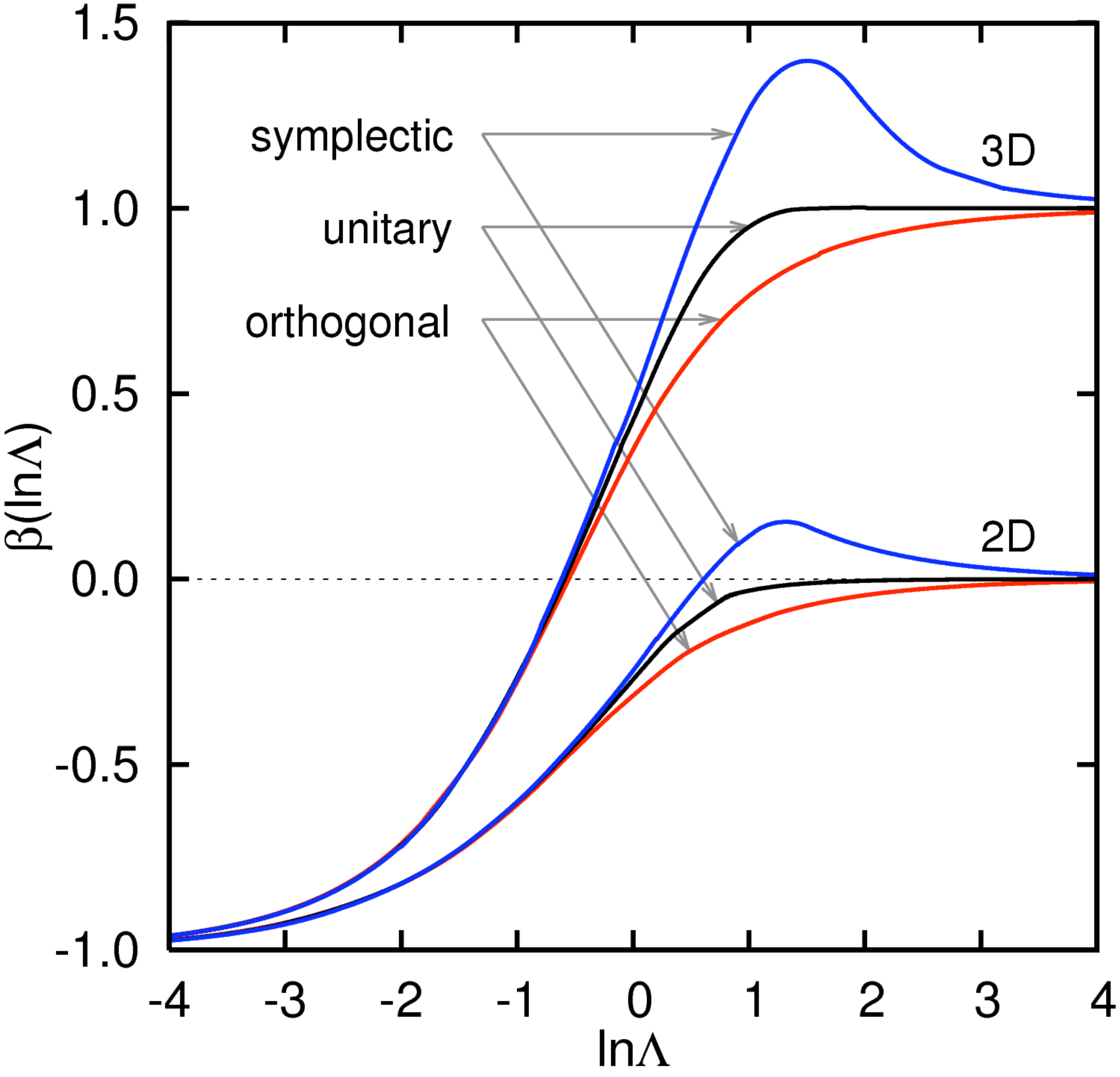
$$E\psi_n = H_n\psi_n + V_{n,n+1}\psi_{n+1} + V_{n,n-1}\psi_{n-1}$$

$$\begin{pmatrix} \psi_{n+1} \\ V_{n+1,n}\psi_n \end{pmatrix} = T_n \begin{pmatrix} \psi_n \\ V_{n,n-1}\psi_{n-1} \end{pmatrix} \quad \boldsymbol{T}_n = \begin{pmatrix} \boldsymbol{V}_{n,n+1}{}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{V}^{\dagger}_{n,n+1} \end{pmatrix} \begin{pmatrix} E\boldsymbol{I} - \boldsymbol{H}_n & -\boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{0} \end{pmatrix}$$



$$\lambda(L,W,E,...) = \lim_{L_z\to\infty} \frac{1}{2L_z} \ln T^{+}T \qquad \Lambda = \frac{\lambda}{L}$$

β function



Asada, Slevin, Ohtsuki, PRB ‘06

Universality

- What justifies the universality of the Anderson transition
  - Weak localization correction $\Delta\sigma = -\frac{e^2}{h} \times \frac{2}{\pi} \log L$
  - Mapping to the nonlinear sigma model

$$Y(\boldsymbol{r}, E, \omega) = \left\langle G^{\mathrm{R}}_{E_+}(0, \boldsymbol{r}) G^{\mathrm{A}}_{E_-}(\boldsymbol{r}, 0) \right\rangle_{\mathrm{imp}}$$

c.f. Kubo formula

$$\longrightarrow \quad Y(\boldsymbol{r}, E, \omega) = \frac{(\pi\rho(E_{\mathrm{F}}))^2}{64} \int DQ$$
$$\times \mathrm{Str}[k(1_8 + \Lambda)(1_8 - \tau_3)Q(\boldsymbol{r})(1_8 - \Lambda)(1_8 - \tau_3)kQ(0)] \exp(-F[Q]) \, ,$$

Non-linear sigma model for disordered electron system (Hikami 80, Efetov 83)

$$F[Q] = \frac{\pi\rho(E_{\mathrm{F}})}{8}\mathrm{Str}\int \mathrm{d}\boldsymbol{r}\left[D_0(\nabla Q)^2 + 2\mathrm{i}(\omega + \mathrm{i}\delta)\mathit{\Lambda} Q\right]$$

$$Q = W + \mathit{\Lambda}(1 - W^2)^{1/2}, W = \begin{bmatrix} 0 & Q_{12} \\ Q_{21} & 0 \end{bmatrix}$$

Many models are reduced to this nonlinear sigma model!

In terms of Lie Algebra

(Altland-Zirnbauer, PRB ’97, for review, Evers-Mirlin RMP’08)

$X = iH$ *X* satisfies Lie algebra→Lie group : U(N)

$H = H_{\mathrm{sym}} + iH_{\mathrm{antisym}}$ Real antisymmetric matrix forms subalgebra

$$[H_{antisym}, H'_{antisym}] \in antisymmetric$$

→SO(N) Lie group→ The real symmetric Hamiltonian (orthogonal class) corresponds to U(N)/O(N)

Discrete symmetry like

$$H = \begin{pmatrix} 0 & h \\ h^\dagger & 0 \end{pmatrix} \qquad \tau_z H \tau_z = -H$$

$X = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$ forms subalgebra $[X, X'] = \begin{pmatrix} h & 0 \\ 0 & h' \end{pmatrix}$

and the group of *H* corresponds to U(2N)/(U(N)xU(N))

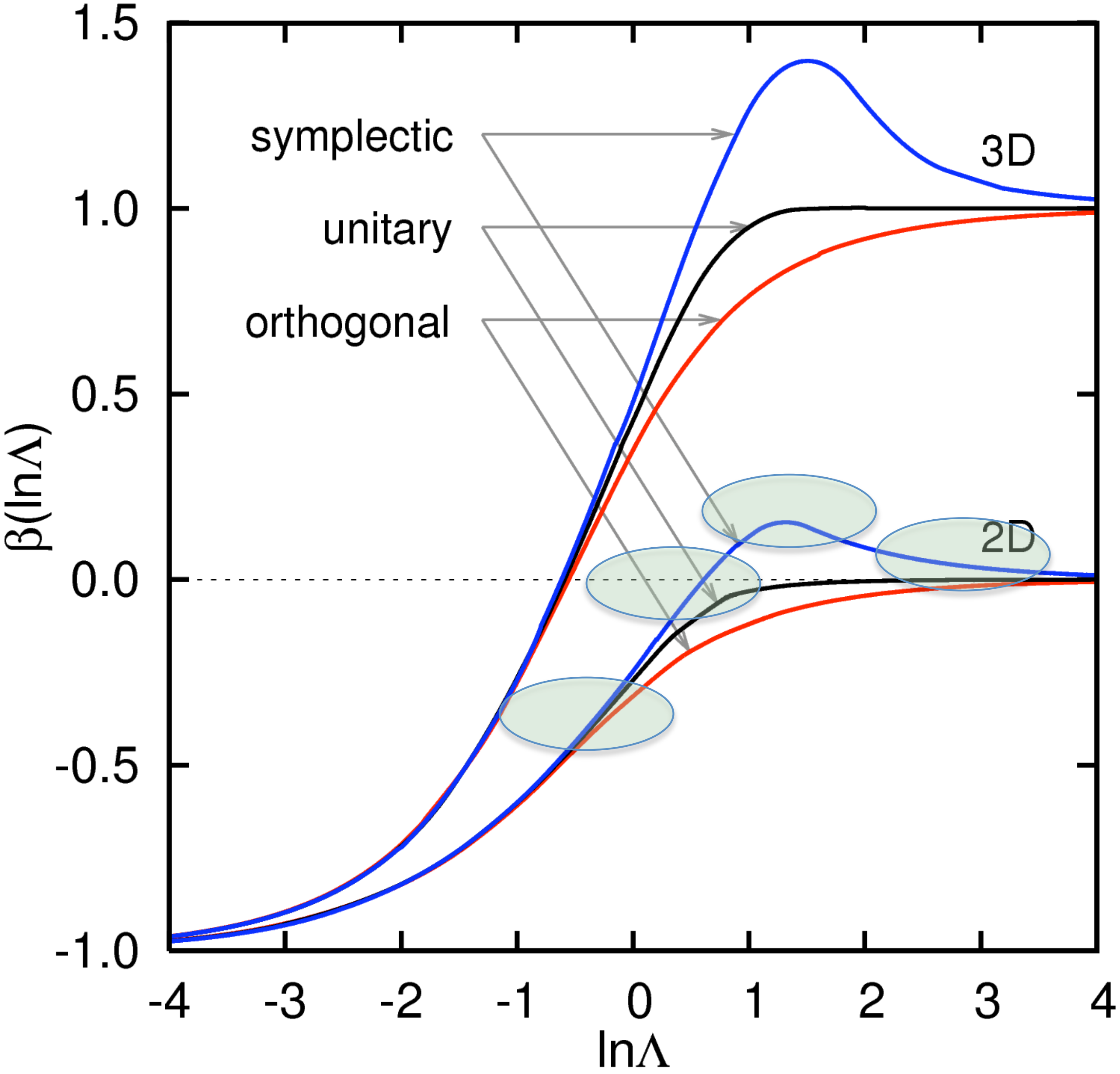
10 universality classes

- Wigner-Dyson class (3 classes)
  - Unitary
  - Orthogonal (TRS+SRS)
  - Symplectic (TRS only), tangent space of U(2N)/Sp(2N)
- chiral class (3 classes)
  - Wigner-Dyson classes + chiral symmetry
- BdG class (4 classes)
  - TRS, SRS

Anderson transition in 2d

- Scaling theory: all states are localized
  - – But there are exceptions（in fact 9 out of 10 universality classes show Anderson transitions）
- Specific properties for 2d Anderson transition
  - – Wigner-Dyson symplectic class for example
    - Lower critical dimension<2
    - In metal phase, logarithmic increase of conductivity
    - Sensitivity to the edge states
      - – Z_2 topological insulator → quantum spin Hall effect; application to quantum computation?
    - Conformal invariance

β function



Asada, Slevin, Ohtsuki, PRB '06

Model Hamiltonian: SU(2) model

$$H = \sum_{i,\sigma} \varepsilon_i c^\dagger_{i\sigma} c_{i\sigma} - \sum_{(i,j),\sigma,\sigma'} R(i;j)_{\sigma\sigma'} c^\dagger_{i\sigma} c_{j\sigma'},$$

Diagonal matrix element; random onsite potential $-\frac{W}{2} < \epsilon_i < \frac{W}{2}$

Nearest neighbor transfer: random spin rotation

$$R(i;j) = \begin{pmatrix} e^{i\alpha_{ij}} \cos \beta_{ij} & e^{i\gamma_{ij}} \sin \beta_{ij} \\ -e^{-i\gamma_{ij}} \sin \beta_{ij} & e^{-i\alpha_{ij}} \cos \beta_{ij} \end{pmatrix}$$

$$P(\beta)\mathrm{d}\beta = \begin{cases} \sin(2\beta)\mathrm{d}\beta & 0 \leqslant \beta \leqslant \frac{\pi}{2} \\ 0 & \text{otherwise}. \end{cases}$$

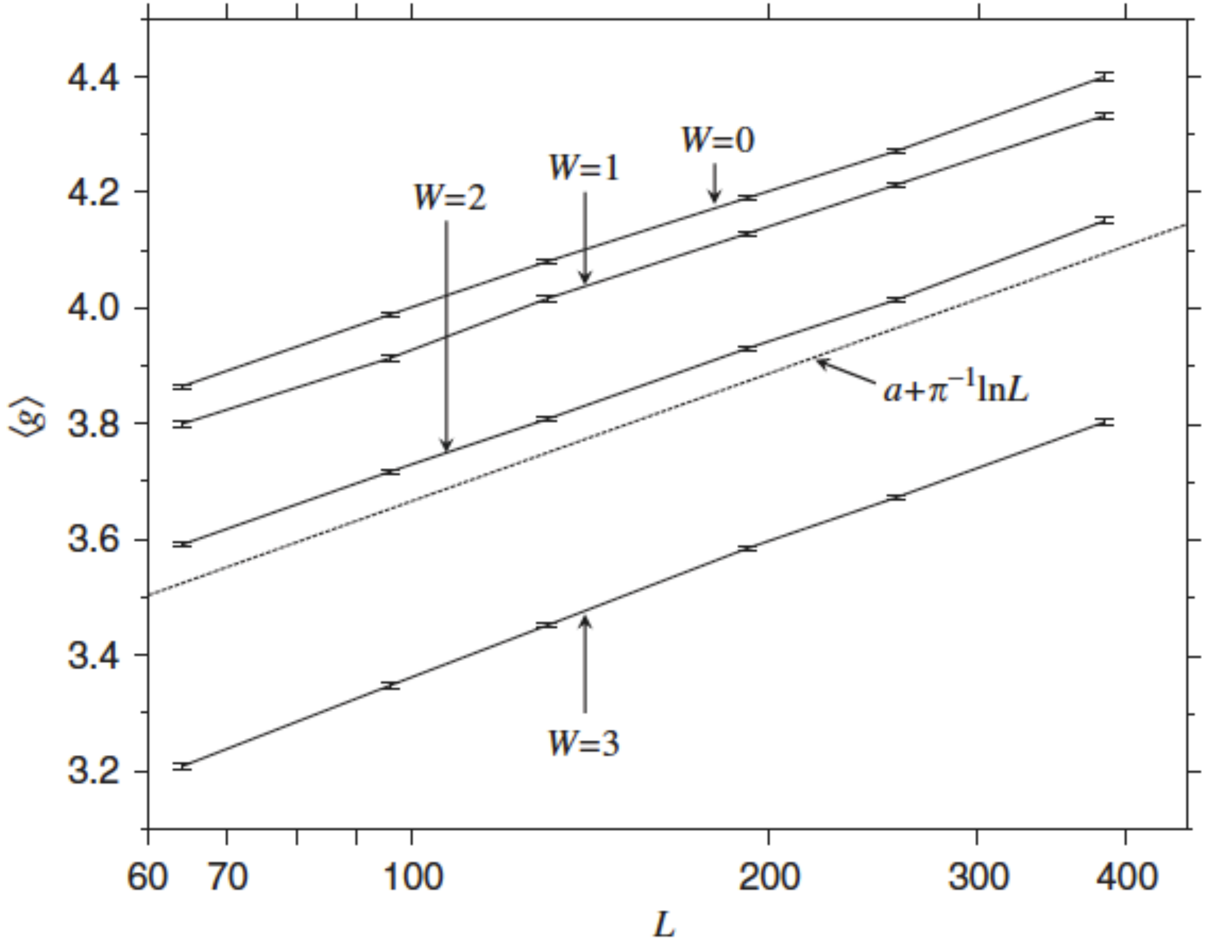
Y. Asada, K. Slevin, and T. Ohtsuki, Phys. Rev. Lett. **89,** 256601 ('02)

Conductance in 2D symplectic systems

- Log(*L*) increase of conductance
  - Hikami, Larkin, Nagaoka theory

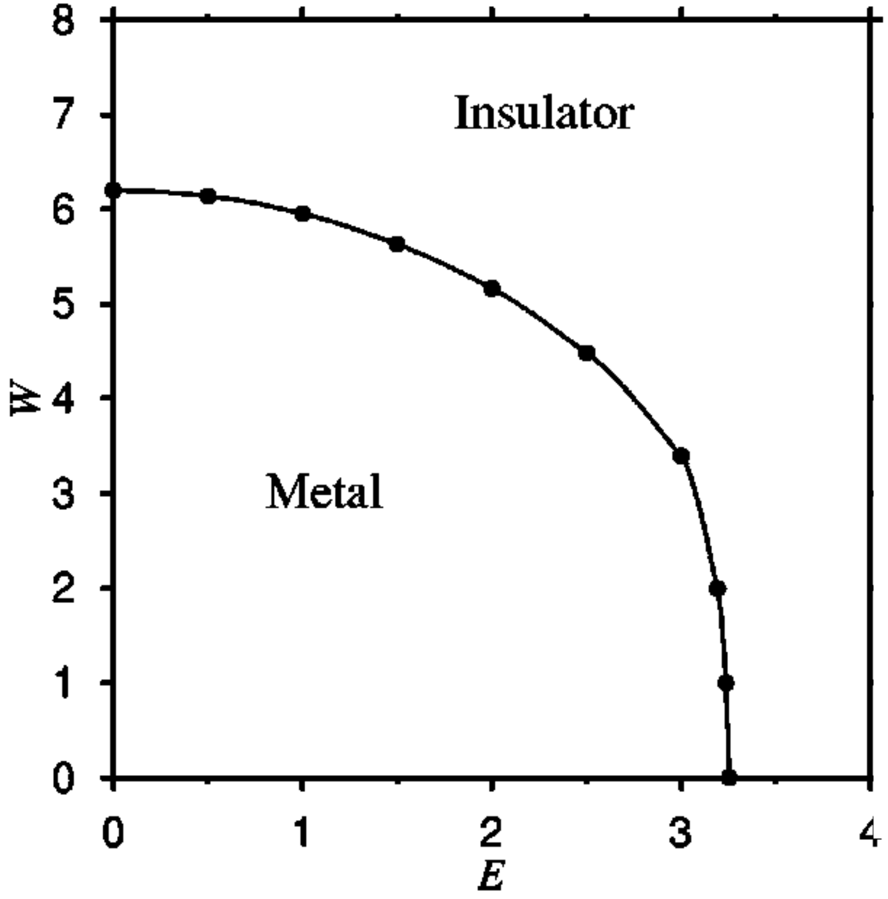
$$g = \sigma = a + \frac{1}{\pi} \ln \left(\frac{L}{\ell} \right)$$

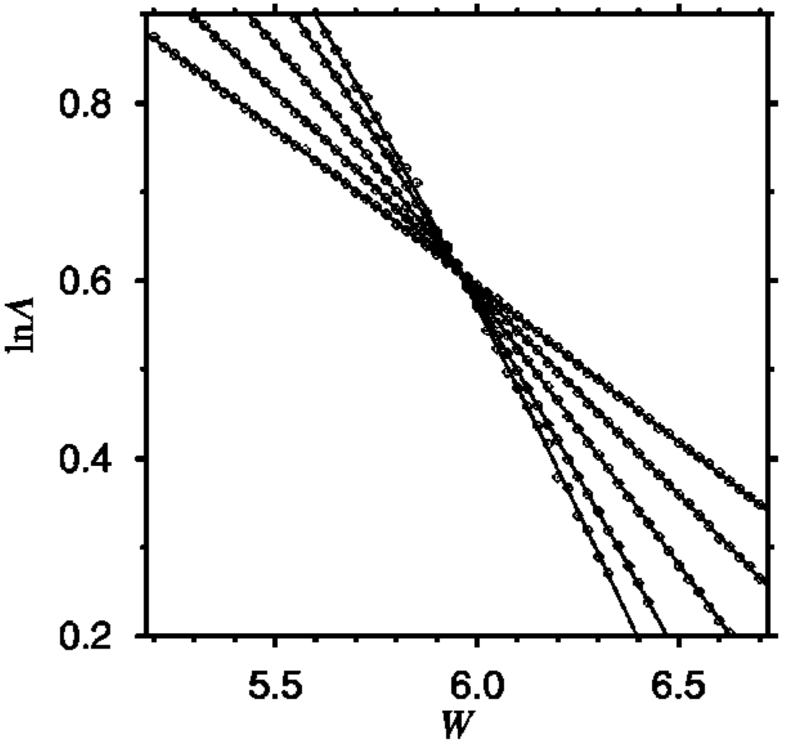
(in units of e^2/h)



Asada et al,. Physica E 34 (2006) 228

Phase diagram for SU(2) model





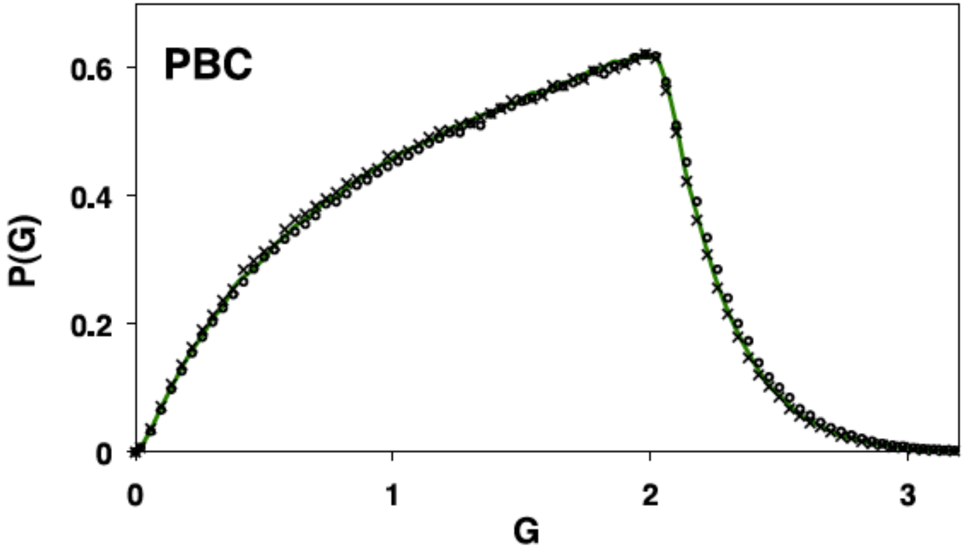
Critical exponent, v=2.75±0.01
Well established for many models

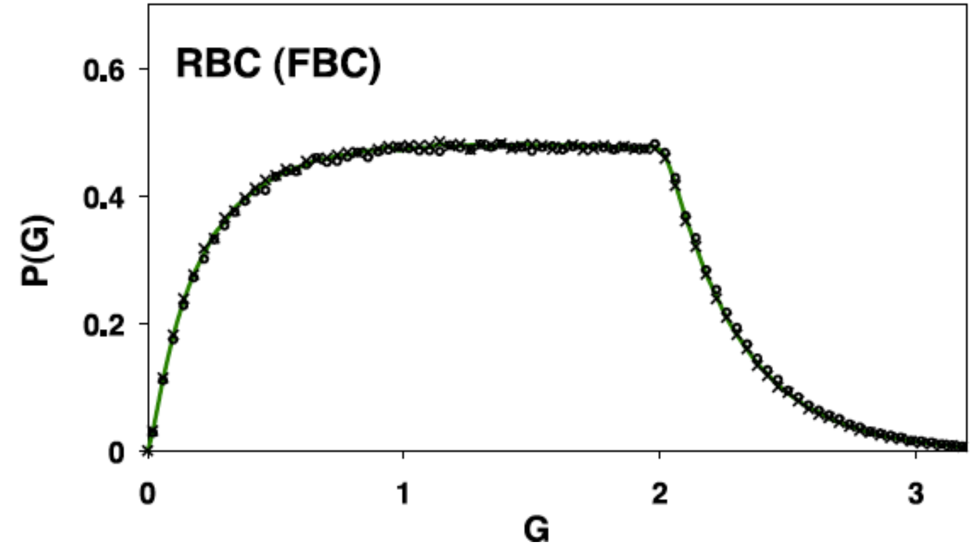
Conductance distribution in 2d

$s = (d-2)\nu$ →0

Scale and model independent

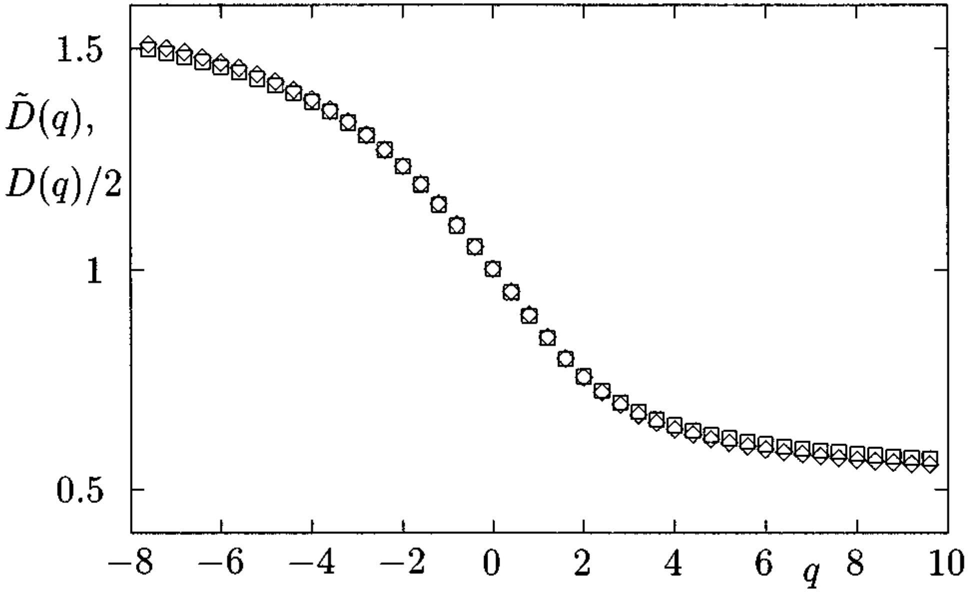
But sensitive to the boundary condition



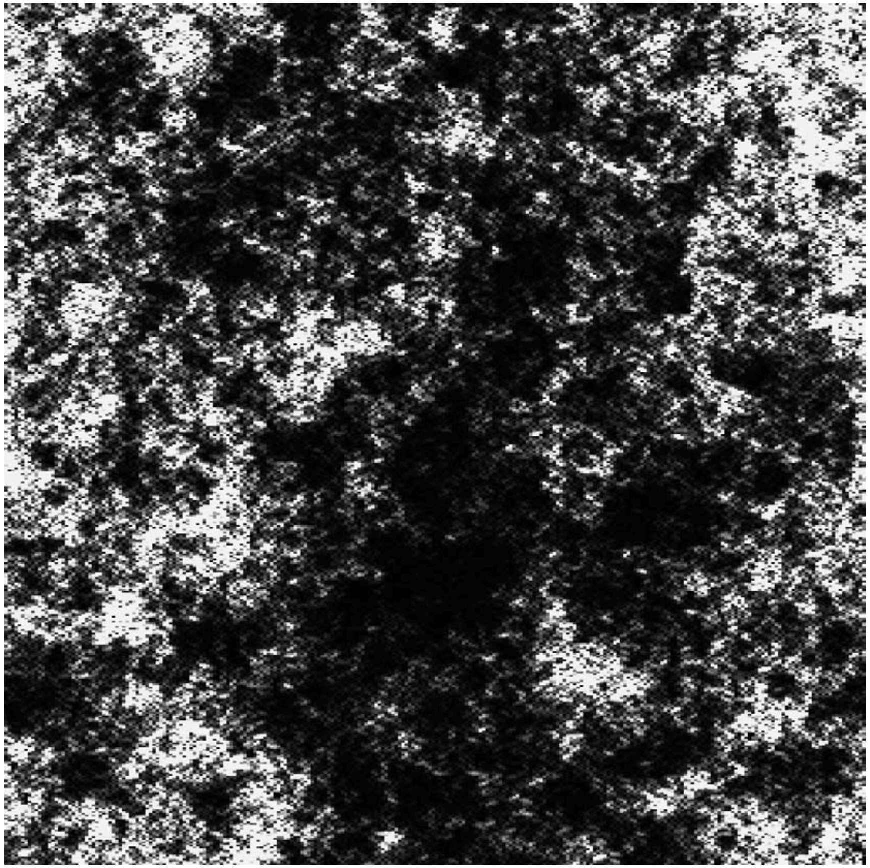


- Wave function at the Anderson transition
    - Scale invariant
    - Multifractal

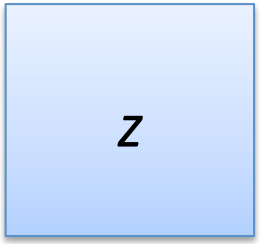
$$p_q(L) := \langle |\Psi(\boldsymbol{r})|^{2q} \rangle \propto \frac{1}{L^{2+(q-1)D(q)}} \propto \frac{1}{L^{2+\tau(q)}}$$



Huckestein, Klesse '97

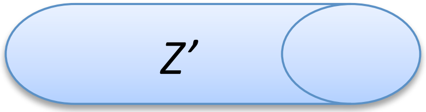


Klesse, Metzler '99



Conformal invariance?

$$\langle \phi(z_1, z_1^*)\phi(z_2, z_2^*)\rangle = |w'(z_1)|^\eta |w'(z_2)^*|^\eta \langle \phi(z_1', z_1'^*)\phi(z_2', z_2'^*)\rangle$$



$$\langle \phi(z_1, z_1^*)\phi(z_2, z_2^*)\rangle \sim |z_1 - z_2|^{-2\eta}$$

$$z' = \frac{L}{2\pi} \ln z \, .$$

L: circumference of cylinder

$$\langle \phi(z_1', z_1'^*)\phi(z_2', z_2'^*)\rangle \sim$$

$$\frac{(2\pi/L)^{2\eta}}{\{2\cosh[2\pi(x_1' - x_2')/L] - 2\cos[2\pi(y_1' - y_2')/L]\}^\eta}$$

$$\rightarrow \exp(-2\pi\eta(x_1' - x_2')/L) = \exp(-(x_1' - x_2')/\xi)$$

$$\xi_{\rm cyl} = \frac{L}{2\pi\eta} \longrightarrow \xi_{\rm cyl} = \frac{L}{\pi(\alpha_0 - 2)}$$

M. Jannsen Phys. Rep. '98

Conformal invariance 2

- Multifractal analysis for 2d $\alpha_0^b = 2.173 \pm 0.001$

  (Obuse et al. PRL ‘07)

- transfer matrix for quasi 1d $\frac{\xi}{L} = \Lambda_{\rm c} = 1.843 \pm 0.01$

  (Asada et al., PRB ’04)

Consistent with the relation

$$\frac{1}{\underline{\Lambda_{\rm c}}} = \underline{\pi(\alpha_0 - 2)}$$

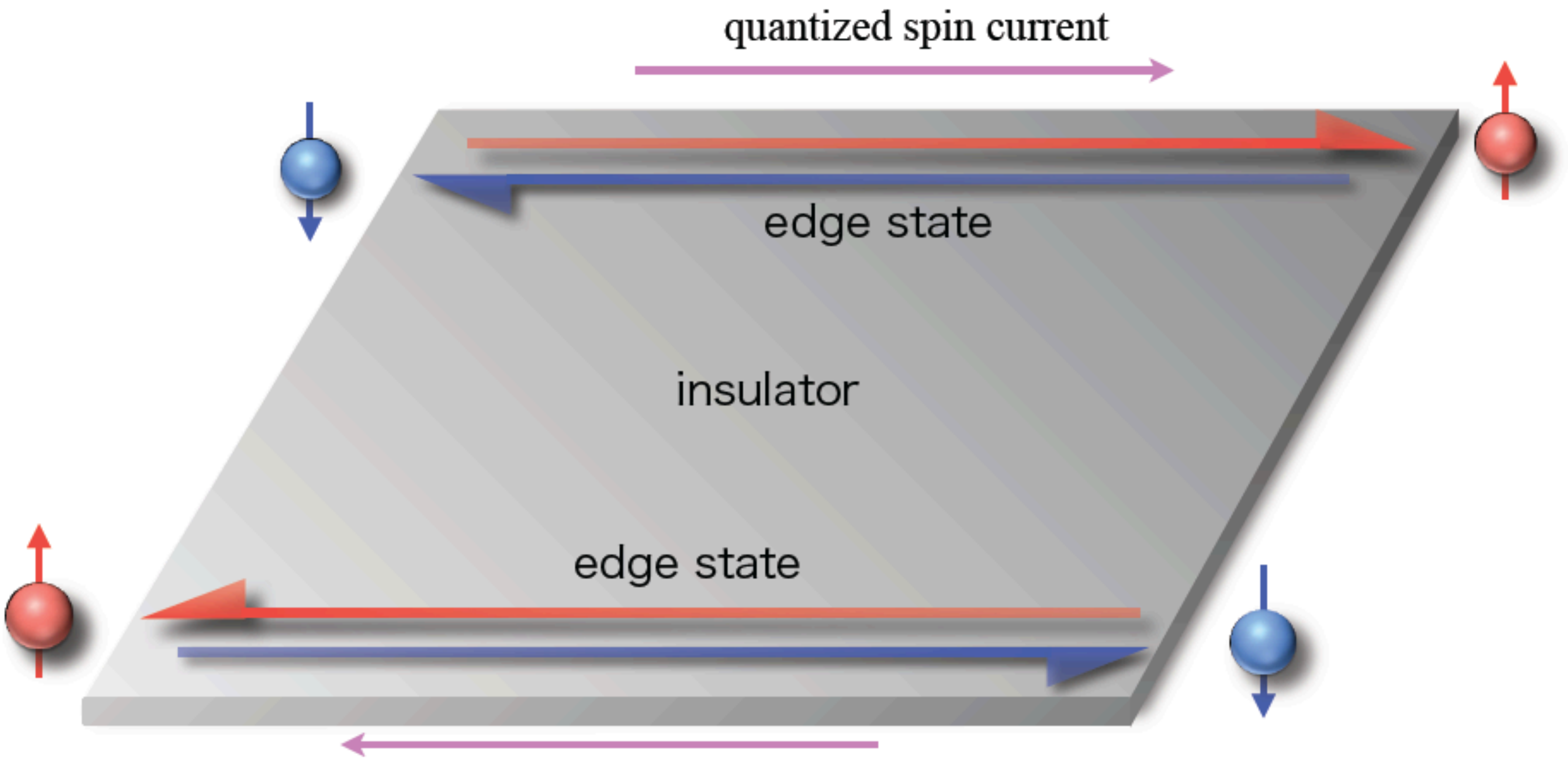
0.544+-0.04

0.5425+-0.0005

- Numerics supports conformal invariance for 2d symplectic class

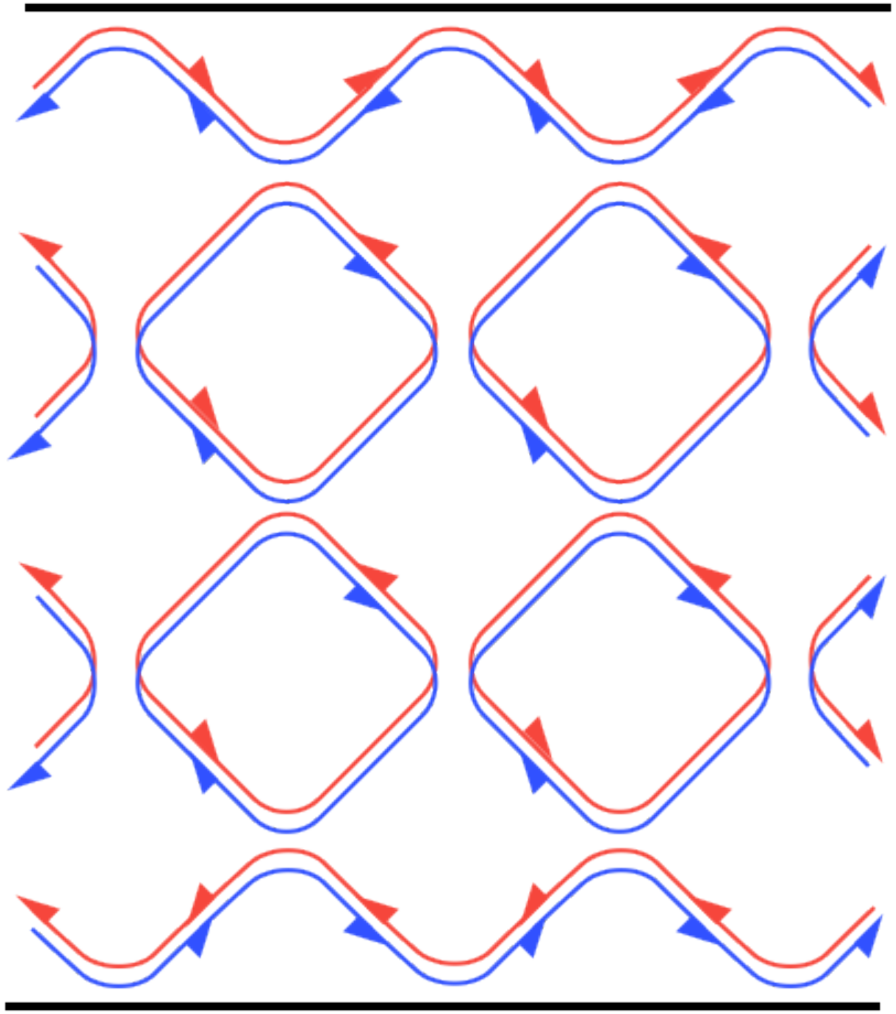
Topological insulator

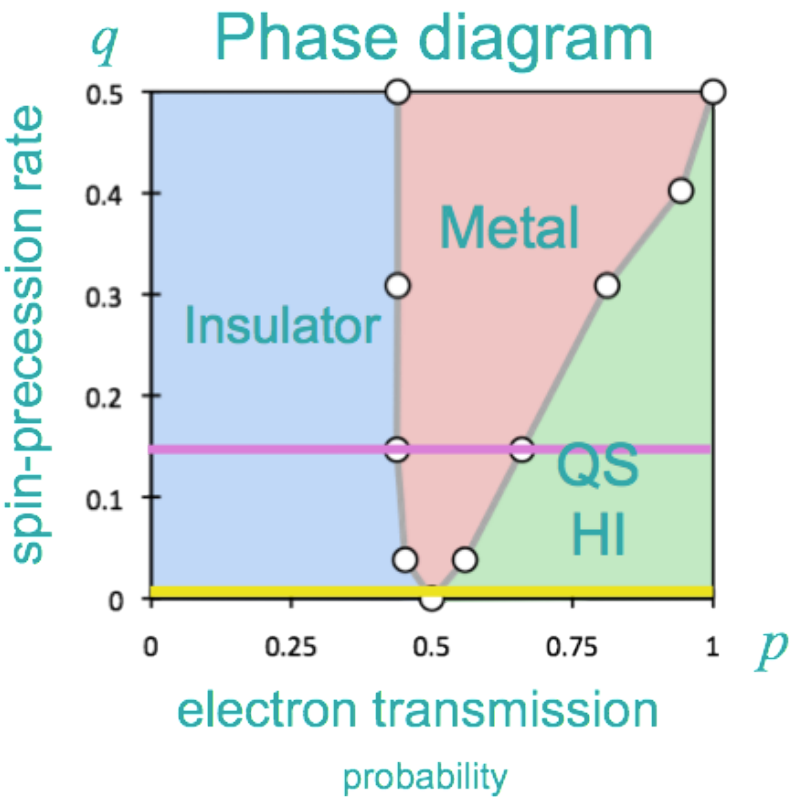
- Quantum spin Hall effect



Network model for QSHE

Obuse, et al. PRB(2007), Avishai, private commun. (2006)

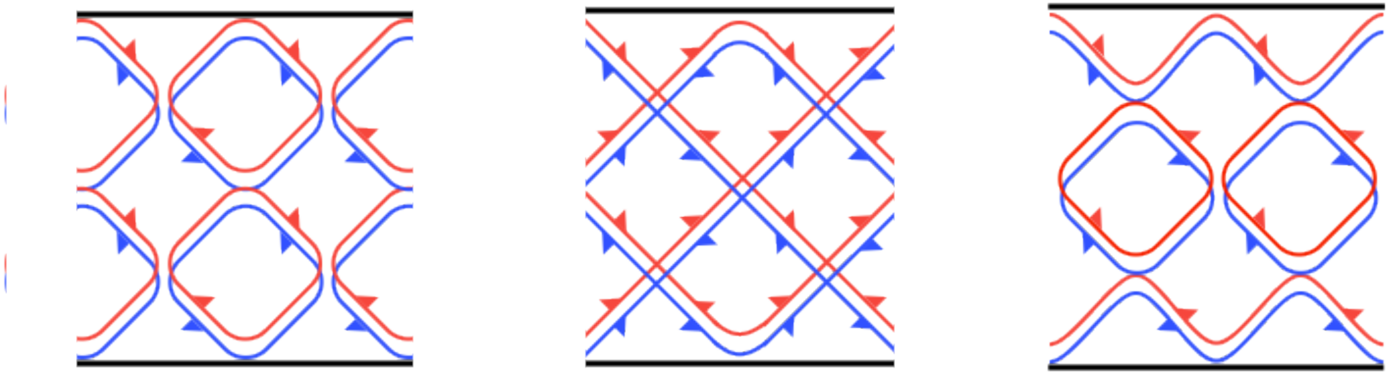




(Fermi energy)

Phases and transition points

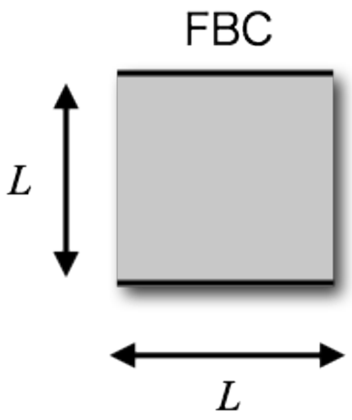
in quantum spin Hall network model



ordinary insulator

metal

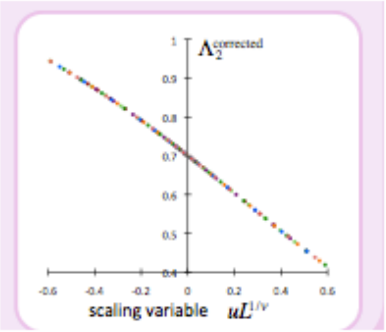
topological insulator



M-OI

M-QSHI

$G = 0$



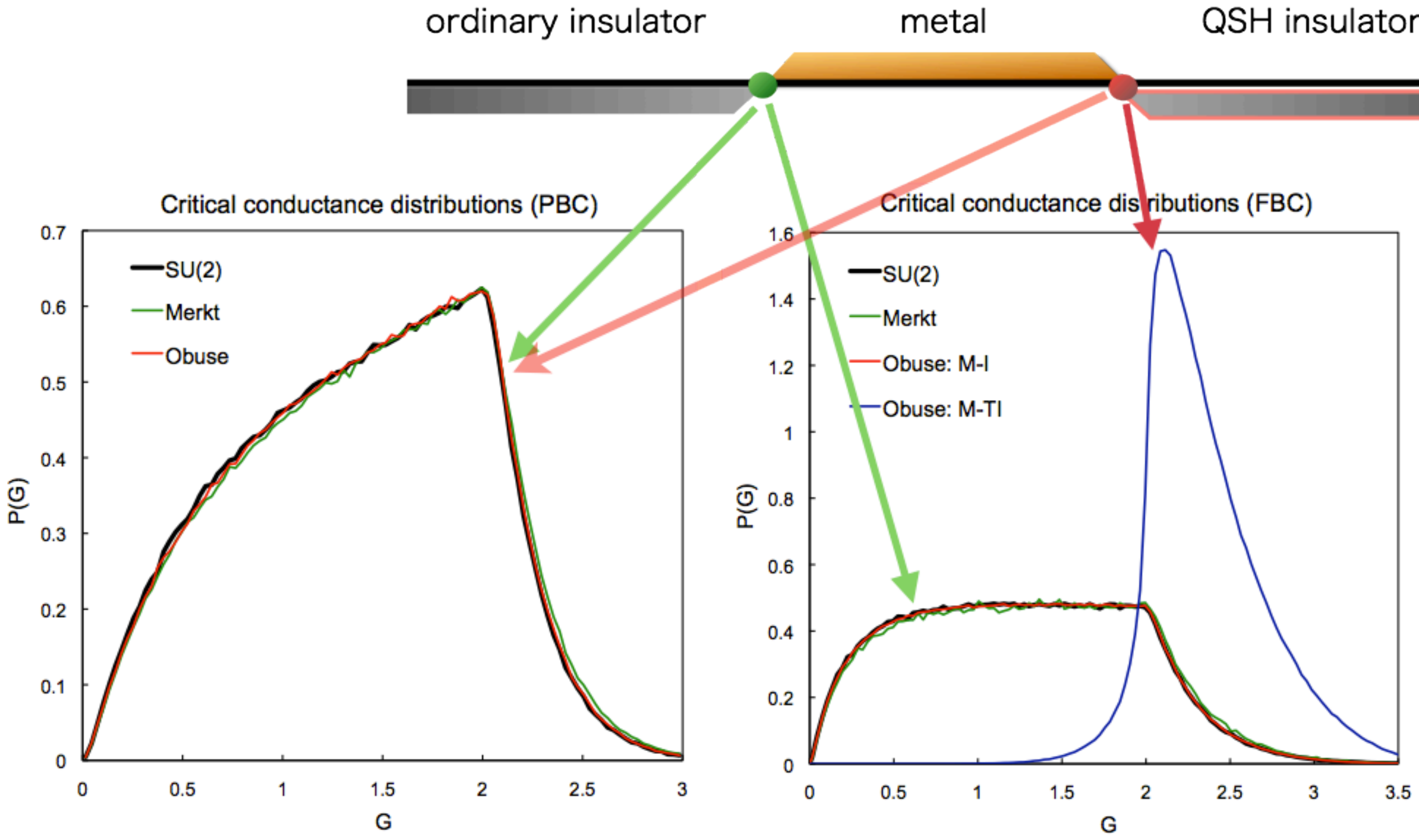
$$G = \underbrace{0}_{\text{ins.}} + \underbrace{1}_{\text{edge}} + \underbrace{1}_{\text{edge}} = 2$$

K. Kobayashi

two-terminal conductance

$$G = \mathrm{Tr}\, \boldsymbol{t}^{\dagger}\boldsymbol{t}\ [e^2/h]$$

Boundary condition dependence and model independence



ordinary insulator

metal

QSH insulator

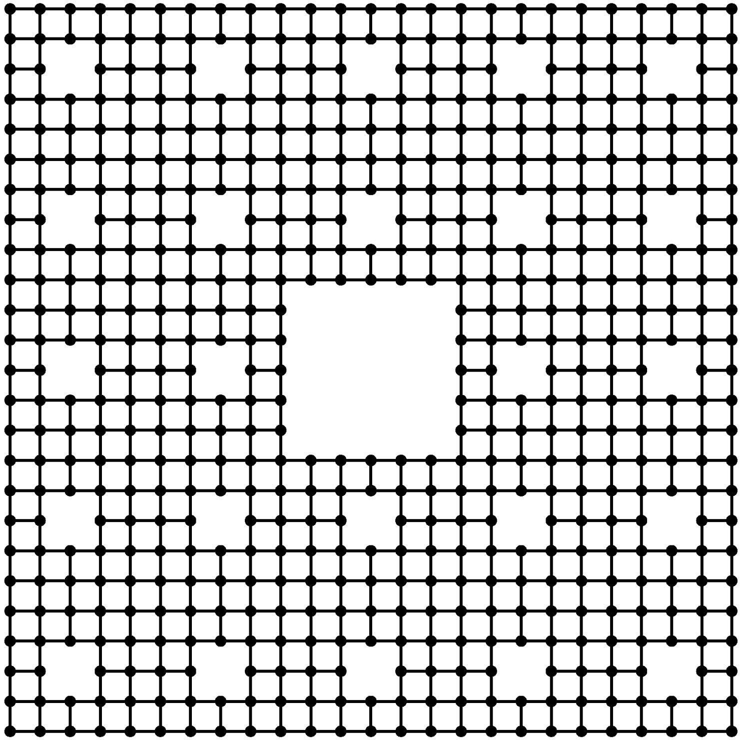
Critical conductance distributions (PBC)

Critical conductance distributions (FBC)

Kobayashi, Ohtsuki, Slevin, Obuse PRB ('10)

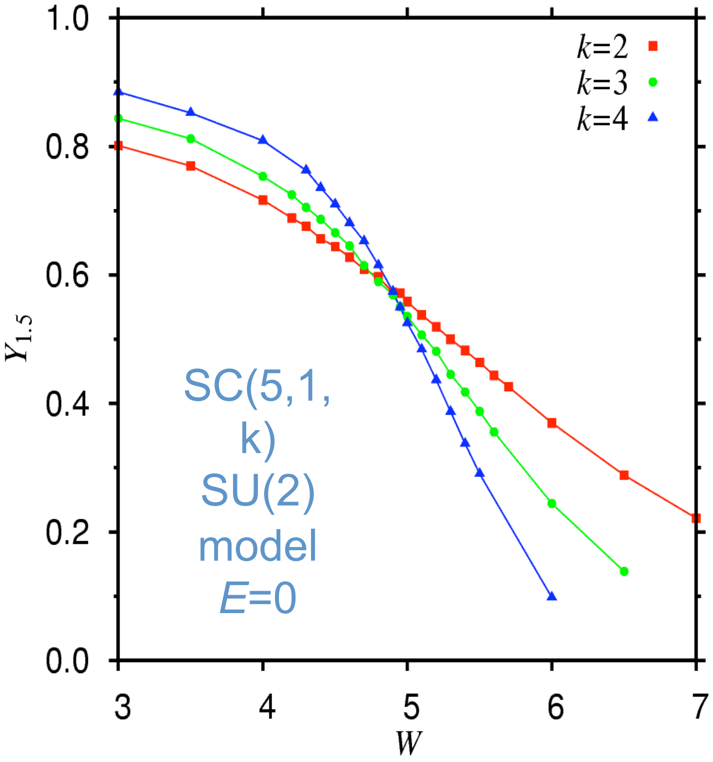
Critical dimensions

- Lower critical dimension d_L<2?
- Analyses based on the level statistics



$d_s = 1.940 \pm 0.009$.

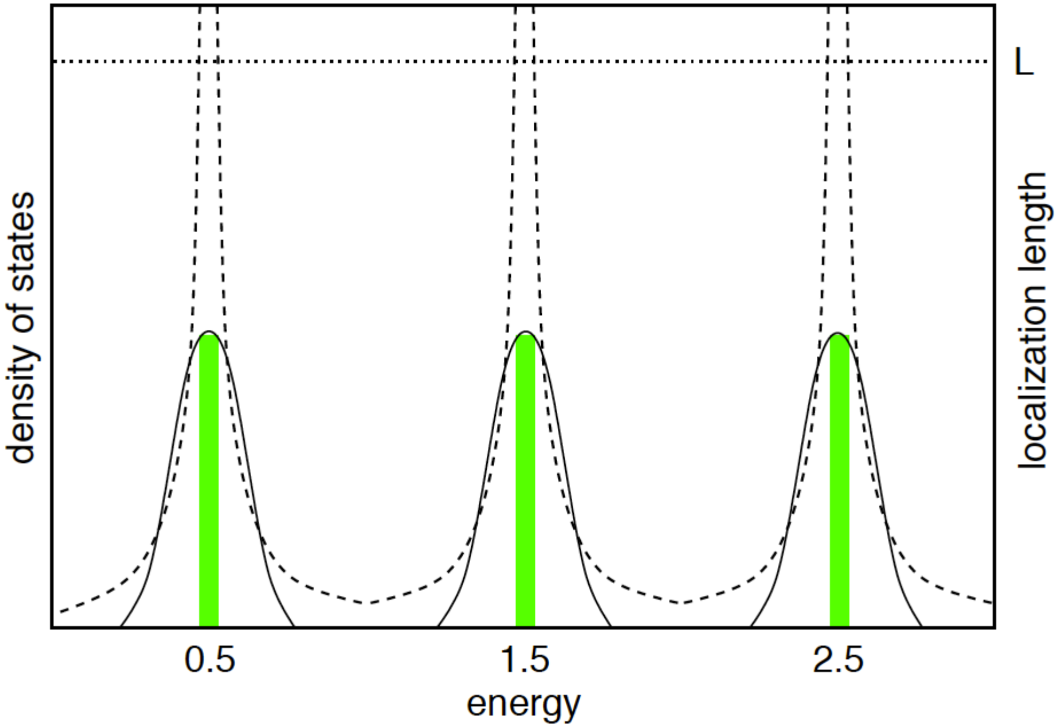
Asada, Slevin, Ohtsuki, PRB '06



$\nu = 3.50 \pm 0.11$

Unitary class

- Standard unitary class has no extended states
- Quantum Hall effect→states are extended at the center of the Landau band



Quantum Hall effect 2

- Critical exponent→still controversial
  - ν=2.37(till 2009)→ν=2.59+-0.01 (Slevin, Ohtsuki, '09)
  - Relation between multifractal exponent and transfer matrix Lyapunov exponent, not well extablished
  - Still much work needed

Summary

- Anderson transition→renewed interested due to recent experimental developments
- Concept of universality classes→useful
- 2d Anderson transition
  - 2d symplectic class
    - Lower critical dimension<2, but what is the value?
    - logarithmic increase of conductance is real?
  - Conformal invariance at the transition is supported
  - Topological insulating phases due to edge states
    - Change of conductance distributions at critical point
  - Unitary class: Quantum Hall effect
    - further studies are needed