

Master Thesis 2009
Weak Values in Quantum Measurement Theory
— Concepts and Applications —

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Contents

A	Uncertainty Relationships	1
A.1	Uncertainty Principle and Uncertainty Relationship	1
A.2	Quantum Mean Square Error	3
A.3	Detection Limit of Gravitational Wave	4
A.4	Heisenberg's Uncertainty Principle Revisited	5
A.5	Ozawa's Inequality	6
A.6	Uncertainty Relationships for Joint Measurement	9
B	Time-Energy Uncertainty Relationships	13
	Bibliography	21

Appendix A

Uncertainty Relationships

In this chapter, we review various uncertainty relationships from the historical view¹. This historical review consists of the derivations and the definitions and the physical meanings.

A.1 Uncertainty Principle and Uncertainty Relationship

The uncertainty principle was initiated by Heisenberg [55]. In the paper [55], Heisenberg explained its physical meaning by the three examples, the gedankenexperiment of gamma-ray microscope gedankenexperiment (position-momentum), the Stern-Gerlach experiment (time-energy) (See App. B.), and the atomic structure (number-phase), and formulated these following the Dirac-Jordan theory [36, 75], which is the non-commutative theory. In the following, we explain the gamma-ray microscope gedankenexperiment (Fig. A.1) [56]. The limits on the accuracy of the location δx of the image is given by

$$\delta x \sim \frac{\lambda}{\sin \epsilon}, \quad (\text{A.1})$$

where λ denotes the wave length of the scattered radiation² and ϵ denotes the half angle of aperture of the object. The direction of the scattered light must then, in principle, be considered as undetermined within this angle ϵ . Hence, according to the Compton effect, the component of the momentum of the material particle in the x -direction is undetermined, after the collision, by an amount

$$\delta p_x \sim \frac{h}{\lambda} \sin \epsilon. \quad (\text{A.2})$$

¹As far as the author knows, there is no inclusive historical review of the uncertain relationships. Over the historical review part, we refer to [104, 69]

² λ can be different from the wave length of the incident radiation.

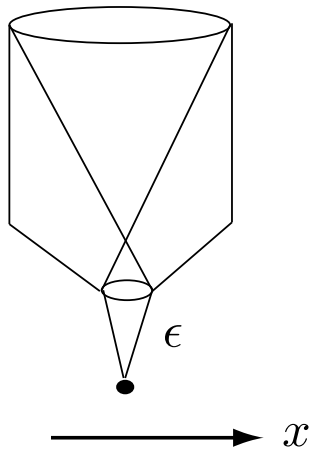


Figure A.1: Heisenberg's gamma-ray gedanken microscope.

From these equations, we obtain

$$\delta x \cdot \delta p_x \sim h. \quad (\text{A.3})$$

This equation³ was based on classical mechanics but he derived it by the Dirac-Jordan theory.

In the last paragraph of the paper [55], Heisenberg added to the pre-publication proof⁴ that Bohr pointed out the direct connection between the uncertain principle and the wave-particle duality. Then, Bohr introduced the complementarity based on the foundations of quantum mechanics. The concept of complementarity is stated that a single quantum mechanical entity can either behave as a particle or as wave, but never simultaneously as both [19].

In the same year to publish the Heisenberg paper, Kennard derived the following inequality [78] as

$$\sigma(x) \cdot \sigma(p) \geq \frac{\hbar}{2}, \quad (\text{A.4})$$

where $\sigma(x)$ and $\sigma(p)$ are the standard deviations of the position and momentum. He considered when measuring some quantity, the amount of this probabilistically changes and we only statistically know the average value as we repeat the measurement. He concluded that this inequality was taken as the Heisenberg uncertainty principle (A.3)⁵

³Almost all physicists interpret the Heisenberg uncertainty principle as Eq. (A.17) or seem to be at cross-purposes with it as Eq. (A.5).

⁴We can get this from the following website.

<http://osulibrary.oregonstate.edu/specialcollections/coll/pauling/bond/papers/corr155.1.html>

⁵In the abstract of the original paper: Das Ergebnis kann dahin formuliert werden, dass der Fällen nur in der Hisenbsergschen Unbestimmtheitsrelation zwischen den Werten kanonisch konjugierter Variablen besteht.

and there occurs the error in classical mechanics on the measurement but the error is the theoretically inevitable quantity in quantum mechanics.

Weyl derived the same inequality (A.4) following the Cauchy-Bunyakowski-Schwarz inequality [142, Appendix 1]. Robertson derived the general inequality for non-commutative observables [118],

$$\sigma(A) \cdot \sigma(B) \geq \frac{|\langle [A, B] \rangle|}{2}, \quad (\text{A.5})$$

which is called *Robertson's uncertainty relationship* and where $\sigma(A)$ and $\langle A \rangle$ are the standard deviation and the average value of the observable A , respectively. Furthermore, Schrödinger derived the following inequality [124];

$$(\sigma(A) \cdot \sigma(B))^2 \geq \left| \frac{\langle [A, B] \rangle}{2} \right|^2 + \left(\frac{\langle \{A, B\} \rangle}{2} - \langle A \rangle \cdot \langle B \rangle \right)^2. \quad (\text{A.6})$$

Summing up Robertson and Schrödinger's works, we conclude that

$$\sigma(A) \cdot \sigma(B) \geq |\langle AB \rangle - \langle A \rangle \langle B \rangle| \geq \frac{|\langle [A, B] \rangle|}{2}. \quad (\text{A.7})$$

The first inequality is due to Schrödinger, and the second to Robertson, which are derived as follows [86].

Since $\langle AB \rangle = \text{Tr}[\sqrt{\rho}AB\sqrt{\rho}]$ is an inner product for $A\sqrt{\rho}$ and $B\sqrt{\rho}$ as the Hilbert-Schmidt operators for the density (Hermitian) operator ρ , the Cauchy-Bunyakowski-Schwarz inequality gives

$$\langle A^2 \rangle \cdot \langle B^2 \rangle \geq |\langle AB \rangle|^2. \quad (\text{A.8})$$

Replacing A by $A - \langle A \rangle$ and B by $B - \langle B \rangle$, we obtain the first inequality. For the Hermitian operators, we obtain

$$|\langle AB \rangle|^2 \geq \frac{1}{4} |\langle \{A, B\} \rangle + \langle [A, B] \rangle|^2 = \frac{1}{4} |\langle \{A, B\} \rangle|^2 + \frac{1}{4} |\langle [A, B] \rangle|^2. \quad (\text{A.9})$$

Deleting the anti-commutator yields the Robertson inequality.

A.2 Quantum Mean Square Error

In classical mechanics and general statistical theory, the definition of the error is almost all taken as the mean square error since Gauss showed its efficiency [46]. Analogously, we introduce the quantum mean square error as follows.

Definition A.1 (Error operator). *For any observable A on the target system \mathcal{H}_s , the error operator for the probe \mathcal{H}_p is*

$$N_A := U(I \otimes M)U^\dagger - A \otimes I := M^{\text{out}} - A^{\text{in}}, \quad (\text{A.10})$$

where U is some evolution operator for the combined system and M is the meter observable for the probe \mathcal{H}_p . Furthermore,

$$n_A := \text{Tr}_{\mathcal{H}_p} N_A \quad (\text{A.11})$$

is called an induced error operator for the observable A .

Definition A.2 (Quantum mean square error). For any state ρ and observable A for the target system \mathcal{H}_s , the quantum mean square error for the probe \mathcal{H}_p is

$$\epsilon(A) := \langle N_A^2 \rangle^{1/2} \quad (\text{A.12})$$

$$= \text{Tr} |(M^{\text{out}} - A^{\text{in}})^2 (\rho \otimes \xi)|^{1/2}, \quad (\text{A.13})$$

where ξ is the probe initial state.

A.3 Detection Limit of Gravitational Wave

The discussion on the detection limit of gravitational waves substantially contributes to understand the Heisenberg uncertainty relationship. A *gravitational wave* is a fluctuation in the curvature of space-time which propagates as a wave, traveling outward from a moving object or system of objects. A *gravitational radiation* is the energy transported by these waves. The existence of the gravitational wave is proven from the Einstein equation by Einstein [38]. As an examples of systems which emit gravitational waves, there are binary star systems, where the two stars in the binary are white dwarfs, neutron stars, and black holes. Although gravitational radiation has not yet been directly detected, it has been indirectly shown to exist by slowing the period of revolution of PSR B1913+16 by 76 microseconds per a year by Hulse and Taylor [67]. Recently, we try to directly detect gravitational waves by the Mach-Zehnder interferometer, e.g. LIGO⁶ and TAMA300⁷. A passing a gravitational wave then slightly stretches one arm as it shortens the other. Therefore, the interference pattern is changed. The problem is how accurate can we decide the length of the arms.

To discuss the problem to measure the mirror, the mirror of the interferometer is taken as a free particle with the mass m and a quantum stuff. At some time, $t = 0$, we measure the position of the mirror. We measure it again at $t = \tau$. Caves *et al.* [27] showed

$$\begin{aligned} [\sigma(x(\tau))]^2 &= [\sigma(x(0))]^2 + \left(\frac{\sigma(p(0))\tau}{m} \right)^2 \geq 2\sigma(x(0)) \cdot \sigma(p(0)) \cdot \frac{\tau}{m} \\ &\geq \frac{\hbar\tau}{m}, \end{aligned} \quad (\text{A.14})$$

⁶Laser Interferometer Gravitational-Wave Observatory.

⁷The 300m Laser Interferometer Gravitational Wave Antenna.

which the bound is called the standard quantum limit (SQL). However, Yuen pointed out that the above inequality was wrong and was proposed as

$$[\sigma(x(\tau))]^2 = [\sigma(x(0))]^2 + \left(\frac{\sigma(p(0))\tau}{m}\right)^2 [\sigma(x(0)) \cdot \sigma(p(0)) + \sigma(p(0)) \cdot \sigma(x(0))] \frac{\tau}{m}, \quad (\text{A.15})$$

using the contractive state [145] and where the third term is negative. While Caves showed that we cannot create the contractive state in the von Neumann model [28], Ozawa showed that the following interaction Hamiltonian gives the contractive state as the ground state;

$$H_{int} = \frac{\pi K}{3\sqrt{3}} \left(2\hat{x} \otimes \hat{P} - 2\hat{p} \otimes \hat{X} + \hat{x}\hat{p} \otimes I - I \otimes \hat{X}\hat{P} \right), \quad (\text{A.16})$$

where K is a coupling constant, which is so large to ignore the individual Hamiltonians, and (\hat{x}, \hat{p}) and (\hat{X}, \hat{P}) are the position and momentum operators on the target system and the probe, respectively [98]. Finally, Maddox judged that Ozawa finally showed the detection limit of this problem and concluded the paper in the hope to explore a new uncertainty relationship⁸ in Nature [87].

A.4 Heisenberg's Uncertainty Principle Revisited

In the previous subsections, we have shown the difference between the error on the measurement the standard derivation of wave packets. Then, Ozawa discussed the Heisenberg uncertainty relationship [101] as

$$\epsilon(A)\eta(B) \geq \frac{|[A, B]|}{2}, \quad (\text{A.17})$$

where $\eta(B)$ is defined as follows.

As analogy to the error (Sec. A.2), we define the quantum mean square disturbance as follows.

Definition A.3 (Disturbance operator). *For any observable B on the target system \mathcal{H}_s , the disturbance operator for the probe \mathcal{H}_p is*

$$D_B := U(B \otimes I)U^\dagger - B \otimes I := B^{\text{out}} - B^{\text{in}}, \quad (\text{A.18})$$

where U is some evolution operator for the combined system $\mathcal{H}_s \otimes \mathcal{H}_p$. Furthermore,

$$d_B := \text{Tr}_{\mathcal{H}_p} D_B \quad (\text{A.19})$$

is called an induced disturbance operator for the observable B .

⁸In the original paper: It is also far from obvious how the particular example on which the conclusion rests can be turned into a realistic measuring equipment that would allow those who design equipment to exploit this recipe for beating SQL. But none of this will damp enthusiasm for overcoming what often seems an intolerable constraint on the freedom to design accurate measuring equipment.

Definition A.4 (Quantum mean square disturbance). *For any state ρ and observable B for the target system \mathcal{H}_s , the quantum mean square disturbance for the probe \mathcal{H}_p is*

$$\eta(A) := \langle D_B^2 \rangle^{1/2} \quad (\text{A.20})$$

$$= \text{Tr} |(B^{\text{out}} - B^{\text{in}})^2 (\rho \otimes \xi)|^{1/2}, \quad (\text{A.21})$$

where ξ is the probe initial state.

A.5 Ozawa's Inequality

Ozawa derived the uncertainty relationship included in the error and disturbance on the measurement and the standard derivation of wave packets as

$$\epsilon(A)\eta(B) + \epsilon(A)\sigma(B) + \sigma(A)\eta(B) \geq \frac{|\langle [A, B] \rangle|}{2}. \quad (\text{A.22})$$

This inequality means to violate the Heisenberg uncertainty relationship (A.17) and is called the *Ozawa inequality* [101].

On deriving the Ozawa inequality, it is useful the following lemma.

Lemma A.1. *Let A and ρ be a target observable and an initial target state on \mathcal{H}_s and $(\mathcal{H}_p, \sigma, U, M)$ be a quadruplet about the measurement, the probe Hilbert space, the initial probe state, the unitary operator on the combined system, and the meter observable. We obtain*

$$\epsilon(A)\eta(B) + \frac{|\langle [N_A, B^{\text{in}}] \rangle|}{2} + \frac{|\langle [A^{\text{in}}, D_B] \rangle|}{2} \geq \frac{|\langle [A, B] \rangle|}{2}. \quad (\text{A.23})$$

Proof. From the definitions of the error and disturbance operators (A.10, A.18), we obtain

$$\begin{aligned} 0 &= [A^{\text{out}}, B^{\text{out}}] \\ &= [A^{\text{in}} + N_A, B^{\text{in}} + D_B] \end{aligned} \quad (\text{A.24})$$

to transform

$$[N_A, D_B] + [N_A, B^{\text{in}}] + [A^{\text{in}}, D_B] = -[A^{\text{in}}, B^{\text{in}}]. \quad (\text{A.25})$$

Taking the average value for the density operator $\rho \otimes \sigma$ on the both sides, we follow the triangle inequality to obtain

$$\begin{aligned} |\langle [N_A, D_B] \rangle| + |\langle [N_A, B^{\text{in}}] \rangle| + |\langle [A^{\text{in}}, D_B] \rangle| &\geq |\langle [A^{\text{in}}, B^{\text{in}}] \rangle| \\ &= \langle [A, B] \rangle. \end{aligned} \quad (\text{A.26})$$

Using the relationship as the above,

$$\epsilon(A) \geq \sigma(N_A), \quad (\text{A.27})$$

$$\eta(B) \geq \sigma(D_B), \quad (\text{A.28})$$

and the Robertson uncertainty relationship (A.5), we obtain

$$\begin{aligned}\epsilon(A) \cdot \eta(B) &\geq \sigma(N_A) \cdot \sigma(D_B) \\ &\geq \frac{\langle [N_A, D_B] \rangle}{2}.\end{aligned}\tag{A.29}$$

Then, we obtain

$$\begin{aligned}\epsilon(A)\eta(B) + \frac{|\langle [N_A, B^{\text{in}}] \rangle|}{2} + \frac{|\langle [A^{\text{in}}, D_B] \rangle|}{2} &\geq \frac{\langle [N_A, D_B] \rangle}{2} + \frac{|\langle [N_A, B^{\text{in}}] \rangle|}{2} + \frac{|\langle [A^{\text{in}}, D_B] \rangle|}{2} \\ &\geq \frac{|\langle [A, B] \rangle|}{2},\end{aligned}\tag{A.30}$$

following the second inequality (A.23), which is the desired inequality. \square

From the left hand side of Eq. (A.22), we obtain

$$\begin{aligned}\epsilon(A)\eta(B) + \epsilon(A)\sigma(B) + \sigma(A)\eta(B) &\geq \epsilon(A)\eta(B) + \sigma(N_A)\sigma(B) + \sigma(A)\sigma(D_B) \\ &\geq \epsilon(A)\eta(B) + \left| \frac{\langle [N_A, B^{\text{in}}] \rangle}{2} + \frac{\langle [A^{\text{in}}, D_B] \rangle}{2} \right| \\ &\geq \epsilon(A)\eta(B) + \frac{|\langle [N_A, B^{\text{in}}] \rangle|}{2} + \frac{|\langle [A^{\text{in}}, D_B] \rangle|}{2} \\ &\geq \frac{|\langle [A, B] \rangle|}{2},\end{aligned}\tag{A.31}$$

following Eqs. (A.10) and (A.18) in the first inequality, the Robertson uncertainty relationship (A.5) in the second inequality, the triangle inequality in the third inequality, and Eq. (A.26) in the fourth inequality.

The case of holding the Heisenberg uncertainty relationship (A.17) is .

$$|\langle [N_A, B^{\text{in}}] \rangle| + |\langle [A^{\text{in}}, D_B] \rangle| = 0.\tag{A.32}$$

In order to characterize a class of measurements satisfying Eq. (A.17), we define that the measurement interaction is said to be *independent intervention* for the pair (A, B) if the noise and the disturbance are independent of the target system; or precisely if there is observables N and D of the probe such that $N_A = I \otimes N$ and $D_B = I \times B$. Under this condition, we obtain

$$\langle [N_A, B^{\text{in}}] \rangle = \langle [A^{\text{in}}, D_B] \rangle = 0.\tag{A.33}$$

Therefore, the Ozawa inequality (A.22) reduces the Heisenberg uncertainty relationship (A.17). The above conclusion was previously suggested in part by Braginsky and Khalili [22, p. 65] with a limited justification and now fully justified⁹.

⁹Generally speaking, the condition that the Ozawa inequality reduces the Heisenberg uncertainty relationship is given by [103, Theorems 6.1 and 6.3].

We consider the case to violate the Heisenberg uncertainty relationship (A.17) to measure the position. We assume that the interaction Hamiltonian as the above (Sec. A.3) be

$$H_{int} = \frac{\pi K}{3\sqrt{3}} \left(2\hat{x} \otimes \hat{P} - 2\hat{p} \otimes \hat{X} + \hat{x}\hat{p} \otimes I - I \otimes \hat{X}\hat{P} \right), \quad (\text{A.34})$$

where K is a coupling constant, which is so large to ignore the individual Hamiltonians, and (\hat{x}, \hat{p}) and (\hat{X}, \hat{P}) are the position and momentum operators on the target system and the probe, respectively. Then, the evolution operator is given by

$$U(t) = \exp \left(-i \frac{H_{int} t}{\hbar} \right). \quad (\text{A.35})$$

We denote the initial probe state as $|\xi\rangle$ and the time to end the measurement interaction as $t = \Delta t$. In the follows, we consider the Heisenberg picture. From the Heisenberg equation, we obtain

$$\begin{aligned} \frac{d}{dt} \hat{x}(t) &= \frac{\pi K}{3\sqrt{3}} [\hat{x}(t) - 2\hat{X}(t)], \\ \frac{d}{dt} \hat{p}(t) &= -\frac{\pi K}{3\sqrt{3}} [\hat{p}(t) + 2\hat{P}(t)], \\ \frac{d}{dt} \hat{X}(t) &= \frac{\pi K}{3\sqrt{3}} [2\hat{x}(t) - \hat{X}(t)], \\ \frac{d}{dt} \hat{P}(t) &= \frac{\pi K}{3\sqrt{3}} [2\hat{p}(t) - \hat{P}(t)], \end{aligned} \quad (\text{A.36})$$

to calculate $\hat{x}(t)$ and $\hat{X}(t)$ as

$$\begin{aligned} \frac{d}{dt} (\hat{x}(t) + \hat{X}(t)) &= \frac{\pi K}{\sqrt{3}} [\hat{x}(t) - \hat{X}(t)], \\ \frac{d}{dt} (\hat{x}(t) - \hat{X}(t)) &= -\frac{\pi K}{3\sqrt{3}} [\hat{x}(t) + \hat{X}(t)]. \end{aligned} \quad (\text{A.37})$$

We calculate

$$\frac{d^2}{dt^2} (\hat{x}(t) + \hat{X}(t)) = - \left(\frac{\pi K}{3} \right)^2 (\hat{x}(t) + \hat{X}(t)) \quad (\text{A.38})$$

to obtain

$$\hat{x}(t) + \hat{X}(t) = \hat{A} \exp \left(i \frac{\pi K}{3} t \right) + \hat{B} \exp \left(-i \frac{\pi K}{3} t \right). \quad (\text{A.39})$$

From the initial condition, $t = 0$, \hat{A} and \hat{B} are determined. By the analogous discussion,

we obtain

$$\begin{aligned}
\hat{x}(t) &= \frac{2}{\sqrt{3}} \left(\sin\left(\frac{\pi}{3}(1+Kt)\right) \hat{x}(0) - \sin\left(\frac{\pi}{3}Kt\right) \hat{X}(0) \right) \\
\hat{p}(t) &= \frac{2}{\sqrt{3}} \left(\sin\left(\frac{\pi}{3}(1-Kt)\right) \hat{p}(0) - \sin\left(\frac{\pi}{3}Kt\right) \hat{P}(0) \right) \\
\hat{X}(t) &= \frac{2}{\sqrt{3}} \left(\sin\left(\frac{\pi}{3}Kt\right) \hat{x}(0) + \sin\left(\frac{\pi}{3}(1-Kt)\right) \hat{X}(0) \right) \\
\hat{P}(t) &= \frac{2}{\sqrt{3}} \left(\sin\left(\frac{\pi}{3}Kt\right) \hat{p}(0) + \sin\left(\frac{\pi}{3}(1+Kt)\right) \hat{P}(0) \right)
\end{aligned} \tag{A.40}$$

Setting $\Delta t = 1/K$, we calculate

$$\begin{pmatrix} \hat{x}(\Delta t) \\ \hat{p}(\Delta t) \\ \hat{X}(\Delta t) \\ \hat{P}(\Delta t) \end{pmatrix} = \begin{pmatrix} \hat{x}(0) - \hat{X}(0) \\ -\hat{P}(0) \\ \hat{x}(0) \\ \hat{p}(0) + \hat{P}(0) \end{pmatrix} \tag{A.41}$$

to obtain the error and disturbance operators as

$$N_x = \hat{Q}(\Delta t) - \hat{x}(0) = 0, \tag{A.42}$$

$$D_p = \hat{p}(\Delta t) - \hat{p}(0) = -\left(\hat{p} \otimes I + I \otimes \hat{P}\right) \tag{A.43}$$

. The quantum mean square error and disturbance are given by

$$\epsilon(x) = \langle N_x^2 \rangle = 0 \tag{A.44}$$

$$\eta(p) = \langle D_p^2 \rangle = \sigma(p) + \sigma(P) + |\langle p \rangle + \langle P \rangle|^2 < \infty. \tag{A.45}$$

Therefore, the Heisenberg uncertainty relations (A.17) can be transformed as

$$\epsilon(x) \cdot \eta(p) = 0. \tag{A.46}$$

This means that the Heisenberg uncertainty relations (A.17) can be violated.

A.6 Uncertainty Relationships for Joint Measurement

We consider joint measurement of non-commutative observables A and B under the unbiased condition, that is,

$$\langle A \rangle = \langle A \rangle_{\text{meas}}, \quad \langle B \rangle = \langle B \rangle_{\text{meas}}, \tag{A.47}$$

hold for the all target states ρ . Ishikawa and Ozawa [70, 71, 100] independently derived the uncertainty relationship between quantum mean square errors as

$$\epsilon(A) \cdot \epsilon(B) \geq \frac{|\langle [A, B] \rangle|}{2}, \quad (\text{A.48})$$

which is called the *Ishikawa-Ozawa inequality*¹⁰. They assumed that the probe consists of macroscopic stuffs to satisfy

$$[U(I \otimes M_A)U^\dagger, U(I \otimes M_B)U^\dagger] = 0, \quad (\text{A.49})$$

where M_A and M_B are meter observables to measure the target observables A and B , respectively. From Eq. (A.49), we obtain that

$$[N_A, N_B] + [N_A, B \otimes I] + [A \otimes I, N_B] + [A, B] \otimes I = 0, \quad (\text{A.50})$$

where the error operator N_A and N_B are defined in Eq. (A.10). From the unbiased condition, we obtain

$$\text{Tr}(n_A \rho) = 0, \quad (\text{A.51})$$

for all ρ to restrict the induced error operator as

$$n_A = 0. \quad (\text{A.52})$$

Then, we obtain

$$\text{Tr}([N_A, B \otimes I](\rho \otimes \xi)) = \text{Tr}([n_A, B]\rho) = 0, \quad (\text{A.53})$$

and similarly $\text{Tr}([N_B, A \otimes I](\rho \otimes \xi)) = 0$, where ξ is the probe initial state. Taking the average of both sides of Eq. (A.50) in the state $\rho \otimes \xi$, we obtain

$$\text{Tr}([N_A, N_B](\rho \otimes \xi)) = -\text{Tr}([A, B]\rho). \quad (\text{A.54})$$

Noting that

$$(\sigma(N_A))^2 = (\epsilon(A))^2 - |\text{Tr}(N_A(\rho \otimes \xi))|^2 \leq (\epsilon(A))^2, \quad (\text{A.55})$$

we obtain

$$\begin{aligned} \epsilon(A) \cdot \epsilon(B) &\geq \sigma(N_A) \cdot \sigma(N_B) \\ &\geq \frac{|\text{Tr}([N_A, N_B](\rho \otimes \xi))|}{2} \\ &= \frac{|\text{Tr}([A, B]\rho)|}{2}, \end{aligned} \quad (\text{A.56})$$

¹⁰Strictly speaking, the inequality derived by Ishikawa [70] is different by the meaning of the error operator.

following the Robertson uncertainty relationship (A.5) in the second inequality, which is desired inequality (A.48). We have applied this to various measurement model (e.g., see [121]).

From the view of the joint measurement, the Ozawa inequality [102] can be analogously derived as

$$\epsilon(A)\epsilon(B) + \epsilon(A)\sigma(B) + \sigma(A)\epsilon(B) \geq \frac{|\langle[A, B]\rangle|}{2}. \quad (\text{A.57})$$

Under the unbiased condition, we have proven that this inequality is reduced to the Ishikawa-Ozawa inequality. Therefore, the Ozawa inequality is substantial under the case without the unbiased condition.

Appendix B

Time-Energy Uncertainty Relationships

In this chapter, we review the papers on the time-energy uncertainty relationships.

Paper 1 (Heisenberg Uncertainty Principle [55]).

$$\epsilon(E)\eta(T) \sim h, \tag{B.1}$$

where $\epsilon(E)$ is defined the error to measure the energy of the system and $\eta(T)$ is defined the time to measure the energy. This relation always keeps when we measure the energy of a specified system.

Einstein posed a paradox during the sixth Solvay conference in 1930 to violate the time-energy uncertainty relationship,

$$\Delta E \Delta t \geq h, \tag{B.2}$$

where ΔE and Δt were defined as uncertainties of the measurement using a box that emits a photon (See Fig. B.1). This is because the box hangs from a spring scale which measures its weight. Its weight is proportional to its rest mass m , hence to its energy E , according to $E = mc^2$. Einstein supposed that we wait for the box to settle down and accurately measure the initial scale. This reading can be as slow and accurate as we like. After the photon leaves the box, we measure the final scale, again as accurate as we like, and from the difference between the two positions we get an accurate measurement of the energy of the emitted photon, that is, $\Delta E < \infty$. On the other hand, the clock in the box tells exactly when the photon was released, that is, $\Delta t = 0$, so we violate Eq. (B.2)¹.

¹ Rosenfeld described the Bohr reaction to this argument. "It was quite a shock for Bohr to be faced with this problem; he did not see the solution at once. During the whole evening, he was extremely unhappy, going from one to the other and trying to persuade them that it could not be true, that it would be the end of physics if Einstein were right; but he could not produce any refutation... The next morning came Bohr's triumph and the salvation of physics..." [105, P.238].

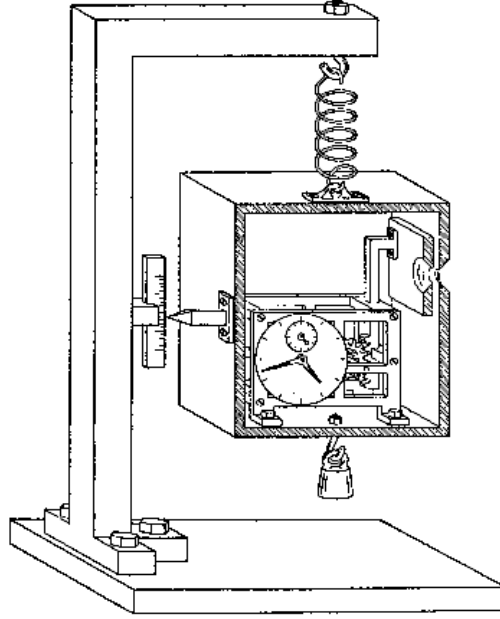


Figure B.1: The clock model proposed by Bohr [20] when Bohr triumphed against Einstein's criticism.

Bohr triumphed as follows. Let x denote the position of the pointer on the scale and p its momentum, with Δx and Δp the corresponding uncertainties. Once we choose Δx , Δp is restricted from the Heisenberg uncertainty relationship, $\Delta x \cdot \Delta p \geq h$, as

$$\frac{h}{\Delta x} \leq \Delta p. \quad (\text{B.3})$$

Bohr assumed the pointer moves after the photon emission. By hanging little weights on the box, we lower it to its original position. When it has returned to its original height, the total weight hanging from it equals the weight of the emitted photon. However, the accurate of this weighting is no better than the smallest added weight $g\Delta m$ that has an observable effect. If we add a mass Δm and wait a time t , the impulsive delivered to the box cannot be greater than $(g\Delta m)t$, which must be greater than Δp to be observable to obtain

$$\Delta p \leq gt\Delta m. \quad (\text{B.4})$$

From Eqs. (B.3) and (B.4), we obtain

$$h \leq gt\Delta x\Delta m = \frac{gt\Delta x\Delta E}{c^2}. \quad (\text{B.5})$$

Einstein assumed that to measure the pointer position could take unlimited time. But Bohr applied a result from general relativity. According to the time-dilation formula of

general relativity, a clock in gravitational field ticks more slowly than a clock in free fall. Two clocks at different heights above the Earth will run at different rates, because of their gravitational potential difference. If the difference in height is Δx , the fractional difference $\Delta t/t$ in their measured times will be

$$\frac{\Delta t}{t} = \frac{g\Delta x}{c^2}. \quad (\text{B.6})$$

If Δx is the uncertainty in the vertical position of a clock, then Δt is the uncertainty in the clock time due to the uncertain gravitational potential. Over a period t , the uncertainty in the time of the clock amounts to

$$\Delta t = \frac{tg\Delta x}{c^2}. \quad (\text{B.7})$$

Combining this result with Eq. (B.5), we obtain

$$\Delta t \Delta E \geq h, \quad (\text{B.8})$$

as required by quantum theory.

Paper 2 (Salecker and Wigner [122]). *Let us consider a linear clock model.*

$$\Delta t_T > \sqrt{\frac{\hbar T}{E}}, \quad (\text{B.9})$$

where $\Delta t_T = \Delta x/\langle v \rangle$ means that a clock measures a time after T , i.e., this is an accuracy. We call the first Salecker-Wigner (S-W) inequality.

$$E > \frac{\hbar}{2\Delta t_T} \sqrt{\frac{T}{\Delta t_T}}. \quad (\text{B.10})$$

We call the second S-W inequality.

We derive the first S-W inequality as follows. From the Heisenberg uncertainty relationship (A.5), we have $\Delta x \Delta p \geq \hbar/2$, where Δx is a variance in the clock position. We assume that the spread in velocity $\Delta v = \Delta p/M$, where M is a mass of the clock, remains. Over time t , the variance of the position grows as

$$\Delta x_t^2 = \Delta x_0^2 + t^2 \Delta v^2 = \Delta x_0^2 + \frac{\hbar^2 t^2}{4M^2 \Delta x_0^2}. \quad (\text{B.11})$$

Fixing the overall time T and minimizing over Δx_0 yields a clock accuracy

$$\Delta x_T \geq \sqrt{\frac{\hbar T}{M}}. \quad (\text{B.12})$$

Since $\langle v \rangle < c$, Eq. (B.12) can be transformed to the desired inequality.

We derive the second S-W inequality as follows. The reading of the clock is connected with the emission of a light signal of duration Δt_T and this imparts to the clock an indeterminate momentum $\hbar/c\Delta t_T$. This momentum would be even greater if a particle of nonzero rest mass were used as a signal. As a result of the emission of the light signal, the velocity of the clock acquires a spread of the amount $\hbar/Mc\Delta t_T$, where M is the mass of the clock. After a further time interval T_2 , it may be at a distance $\hbar T_2/Mc\Delta t_T$ from the point where it would have been without having been read. Therefore, the actual distance between the two points in space time, at the first of which the clock read T_1 less than at the time of the emission of the signal, at the second of which it reads T_2 more than at the time of the emission of the light signal, is

$$\sqrt{(T_1 + T_2')^2 - \left(\frac{\hbar T_2}{Mc^2 \Delta t_T}\right)^2} \sim T_1 + T_2' - \frac{\hbar^2 T_2^2}{2M^2 c^4 (\Delta t_T)^2 (T_1 + T_2')}, \quad (\text{B.13})$$

where

$$T_2' = \sqrt{T_2^2 + \left(\frac{\hbar T_2}{Mc^2 \Delta t_T}\right)^2}. \quad (\text{B.14})$$

Hence, the actual distance (B.13) differs from the time difference $T = T_1 + T_2$ shown by the clock, in the approximation considered, by

$$-\frac{T_1 T_2}{2(T_1 + T_2)} \left(\frac{\hbar T_2}{Mc^2 \Delta t_T}\right)^2. \quad (\text{B.15})$$

The inaccuracy of the clock will be within the limit Δt_T if Eq. (B.15) is less than Δt_T . If one considers the first order to be of the order of magnitude T , we obtain

$$M > \frac{\hbar}{c^2 \Delta t_T} \sqrt{\frac{T}{\Delta t_T}}, \quad (\text{B.16})$$

which is corresponded to the desired equation.

Summing up the two S-W inequalities, the first inequality is restricted from the constant light speed c and the second one is restricted from the causality.

Paper 3 (Margolus and Levitin [89]).

$$\Delta t \geq \frac{\pi \hbar}{2E}, \quad (\text{B.17})$$

where Δt is defined as the time which can move from one state to an orthogonal state with a fixed average energy, denoted as E .

This relation is derived as follows. An arbitrary quantum state can be written as a superposition of energy eigenstates

$$|\psi_0\rangle = \sum_n c_n |E_n\rangle. \quad (\text{B.18})$$

Note that we assume that a system has a discrete spectrum $\{E_n\}$ and choose the ground energy zero so that $E_0 = 0$. If $|\psi_0\rangle$ is evolved for a time t , then it becomes

$$|\psi_t\rangle = \sum_n c_n e^{-iE_n t/\hbar} |E_n\rangle. \quad (\text{B.19})$$

In order to judge the orthogonality for these states, $|\psi_0\rangle$ and $|\psi_t\rangle$, we let

$$S(t) \equiv \langle \psi_0 | \psi_t \rangle = \sum_n |c_n|^2 e^{-iE_n t/\hbar}. \quad (\text{B.20})$$

Since we want to solve the time which can move from one state to an orthogonal state, we want to find the smallest value of t such that $S(t) = 0$. To do this, we note that

$$\begin{aligned} \text{Re}(S(t)) &= \sum_n |c_n|^2 \cos\left(\frac{E_n t}{\hbar}\right) \\ &\geq \sum_n |c_n|^2 \left(1 - \frac{2}{\pi} \left(\frac{E_n t}{\hbar} + \sin\left(\frac{E_n t}{\hbar}\right)\right)\right) \\ &= 1 - \frac{2E}{\pi\hbar} t + \frac{2}{\pi} \text{Im}(S(t)), \end{aligned} \quad (\text{B.21})$$

where we have used the inequality $\cos x \geq 1 - (2/\pi)(x + \sin x)$ for $x \geq 0$. On $S(t) = 0$, both $\text{Re}(S(t)) = 0$ and $\text{Im}(S(t)) = 0$, and so Eq. (B.21) becomes

$$0 \geq 1 - \frac{2E}{\pi\hbar} t. \quad (\text{B.22})$$

Then we obtain the desired inequality.

Paper 4 (Anandan and Aharonov [14], Vaidman [138]).

$$\Delta t \geq \frac{\pi\hbar}{2\Delta E}, \quad (\text{B.23})$$

where Δt is defined as the time which can move from one state to an orthogonal state and ΔE is defined as the variance of the energy.

In this thesis, Vaidman's proof [138] is as follows. In general, for a given observable A , we can decompose

$$A|\psi\rangle = \alpha|\psi\rangle + \beta|\psi_\perp\rangle, \quad (\text{B.24})$$

where $|\psi_\perp\rangle$ is orthogonal to $|\psi\rangle$ and β is real and non-negative. Then $\langle A \rangle \equiv \langle \psi|A|\psi \rangle = \langle \psi|(\alpha|\psi\rangle + \beta|\psi_\perp\rangle)$ yields $\alpha = \langle A \rangle$, and $\langle \psi|A^\dagger A|\psi \rangle = (\alpha^*|\psi\rangle + \beta^*|\psi_\perp\rangle)(\alpha|\psi\rangle + \beta|\psi_\perp\rangle)$ yields $\beta = \sqrt{\langle A^2 \rangle - \langle A \rangle^2} \equiv \Delta A$ to obtain

$$A|\psi\rangle = \langle A \rangle|\psi\rangle + \Delta A|\psi_\perp\rangle. \quad (\text{B.25})$$

From this and the Schrödinger equation, we obtain

$$\frac{d}{dt}|\psi(t)\rangle = -\frac{i}{\hbar}H|\psi(t)\rangle = -\frac{i}{\hbar}(\langle E \rangle|\psi(t)\rangle + \Delta E|\psi_\perp(t)\rangle), \quad (\text{B.26})$$

where $\langle \psi(t)|\psi_\perp(t) \rangle = 0$. Furthermore, we calculate

$$\frac{d}{dt}|\langle \psi(t)|\psi(0) \rangle|^2 = 2\text{Re} \left(\langle \psi(t)|\psi(0) \rangle \langle \psi(0)|\frac{d}{dt}|\psi(t)\rangle \right). \quad (\text{B.27})$$

From these, we obtain

$$\frac{d}{dt}|\langle \psi(t)|\psi(0) \rangle|^2 = -\frac{2\Delta E}{\hbar}\text{Re}(i\langle \psi(t)|\psi(0) \rangle \langle \psi(0)|\psi_\perp(t) \rangle). \quad (\text{B.28})$$

Furthermore, we expand the initial state $|\psi(0)\rangle$ as

$$|\psi(0)\rangle = \langle \psi(t)|\psi(0) \rangle|\psi(t)\rangle + \langle \psi_\perp(t)|\psi(0) \rangle|\psi_\perp(t)\rangle + \alpha|\psi_{\perp\perp}(t)\rangle, \quad (\text{B.29})$$

where $\langle \psi(t)|\psi_{\perp\perp}(t) \rangle = 0$ and $\langle \psi_\perp(t)|\psi_{\perp\perp}(t) \rangle = 0$. The normalization of the quantum states, then, requires that

$$|\langle \psi(0)|\psi_\perp(t) \rangle|^2 = 1 - |\langle \psi(t)|\psi(0) \rangle|^2 - |\alpha|^2. \quad (\text{B.30})$$

Therefore, the maximum value of $|\langle \psi(0)|\psi_\perp(t) \rangle|$ is obtained for $\alpha = 0$, and it is equal to $\sqrt{1 - |\langle \psi(0)|\psi(t) \rangle|^2}$. Thus, the maximum possible absolute value of the rate of change of the square of the overlap is

$$\frac{2\Delta E}{\hbar}|\langle \psi(0)|\psi(t) \rangle|\sqrt{1 - |\langle \psi(0)|\psi(t) \rangle|^2}. \quad (\text{B.31})$$

We find that this maximum rate, indeed, depends only on the value of the overlap and on the energy uncertainty. Therefore, the condition for the fastest evolution to an orthogonal state is that during the whole period of the evolution the right-hand side of Eq. (B.28) is equal to minus Eq. (B.31):

$$\frac{d}{dt}|\langle \psi(t)|\psi(0) \rangle|^2 = -\frac{2\Delta E}{\hbar}|\langle \psi(0)|\psi(t) \rangle|\sqrt{1 - |\langle \psi(0)|\psi(t) \rangle|^2}, \quad (\text{B.32})$$

After introducing a parameter ϕ , $\cos \phi = |\langle \psi(0) | \psi(t) \rangle|$, Eq. (B.32) becomes

$$\frac{d}{dt}\phi = \frac{\Delta E}{\hbar}. \quad (\text{B.33})$$

Since the orthogonal state corresponds to $\phi = \pi/2$, the minimum time is, indeed

$$\Delta t = \frac{\pi}{2\dot{\phi}} = \frac{h}{4\Delta E}. \quad (\text{B.34})$$

The relationship is desired.

Paper 5 (Mandelstam and Tamm [88]).

$$\Delta H \Delta T \geq \frac{\hbar}{2}, \quad (\text{B.35})$$

where ΔH is defined as an uncertainty of the energy and ΔT is defined as the shortest time, during which the average value of a certain quantity is changed by an amount equal to the standard deviation of this quantity.

This equation is derived as the follows. Let R and S denote any two physical quantities and at the same time the corresponding symmetric operators. We have shown that using the Cauchy-Schwartz inequality,

$$\Delta S \Delta R \geq \frac{1}{2} |\langle RS - SR \rangle|, \quad (\text{B.36})$$

where ΔS and ΔR are the standard deviations of the quantities S and R and $\langle \cdot \rangle$ denotes as the average value, and the Heisenberg equation as

$$\hbar \frac{\partial \langle R \rangle}{\partial t} = i (\langle HR - RH \rangle). \quad (\text{B.37})$$

Putting in (B.36) $S \equiv H$, we obtain the inequality,

$$\Delta H \Delta R \geq \frac{\hbar}{2} \left| \frac{\partial \langle R \rangle}{\partial t} \right|. \quad (\text{B.38})$$

The absolute value of an integral cannot exceed th integral of the absolute value of the integrand. Hence, integrating (B.38) from t to $t + \Delta t$ and taking into account that ΔH is constant one gets

$$\Delta H \Delta t \geq \frac{\hbar}{2} \frac{|\langle R_{t+\Delta t} \rangle - \langle R_t \rangle|}{\Delta \langle R \rangle}. \quad (\text{B.39})$$

This relation is desired.

Paper 6 (Aharonov and Bohm [4]).

$$\Delta H \Delta T \geq \frac{\hbar}{2}, \quad (\text{B.40})$$

where ΔH is defined as an uncertainty of the energy and ΔT is defined as the shortest time, during which the average value of a certain quantity is changed by an amount equal to the standard deviation of this quantity. Aharonov and Bohm basically showed that Mandelstam and Tamm bound is universal as follows.

They define the "clock" operator,

$$\hat{T}_c = \frac{1}{2} \left\{ \hat{x}, \frac{1}{\hat{p}} \right\}, \quad (\text{B.41})$$

noting that this operator is singular. When the "clock" Hamiltonian is given $\hat{H}_c = \hat{p}^2/2M$, we obtain the commutation relations,

$$[\hat{H}_c, \hat{T}_c] = i\hbar. \quad (\text{B.42})$$

Essentially, there are three criteria on the time-energy relationship but meanings of Δt are different. About Margolus and Levitin's paper and Anandan and Aharonov's paper, this situation is that a certain state moves to the orthogonal state, i.e., we can take two states with perfect distinguishable. Its minimum time is denoted as Δt . About Mandelstam and Tamm's paper, this situation is that a certain state moves to the state such that we can take two states with distinguishable for a given observable. Note that these situations restrict the Hamiltonian each other.

Bibliography

- [1] E. S. Abers and B. W. Lee, Phys. Rep. **9**, 1 (1973).
- [2] Y. Aharonov, D. Z. Albert, and L. Vaidman, Phys. Rev. Lett. **60**, 1351 (1988).
- [3] Y. Aharonov, P. G. Bergmann, and J. L. Lebowitz, Phys. Rev. **134**, B1410 (1964).
- [4] Y. Aharonov and D. Bohm, Phys. Rev. **122**, 1649 (1961).
- [5] Y. Aharonov and A. Botero, Phys. Rev. A **72**, 052111 (2005).
- [6] Y. Aharonov, S. Popescu, D. Rohrlich, and L. Vaidman, Phys. Rev. A **48**, 4084 (1993).
- [7] Y. Aharonov, S. Popescu, J. Tollaksen, and L. Vaidman, arXiv:0712.0320.
- [8] Y. Aharonov and D. Rohrlich, *Quantum Paradoxes* (Wiley-VCH, Weinheim, 2005).
- [9] Y. Aharonov and J. Tollaksen, to appear in *Visions of Discovery: New Light on Physics Cosmology and Consciousness*, edited by R. Y. Chiao, M. L. Cohen, A. J. Leggett, W. D. Phillips, and C. L. Harper, Jr. (Cambridge University Press, Cambridge, in press), arXiv:0706.1232.
- [10] Y. Aharonov and L. Vaidman, Phys. Rev. A **41**, 11 (1990).
- [11] Y. Aharonov and L. Vaidman, J. Phys. A **24**, 2315 (1991).
- [12] Y. Aharonov and L. Vaidman, in *Time in Quantum Mechanics*, Vol. 1, edited by J. G. Muga, R. Sala Mayato, and I. L. Egusquiza (Springer, Berlin Heidelberg, 2008) pp. 399-447.
- [13] N. I. Akhiezer and I. M. Glazman, *Theory of Linear Operators in Hilbert Space* (Dover, New York, 1993).
- [14] J. Anandan and Y. Aharonov, Phys. Rev. Lett. **65**, 1697 (1990).
- [15] A. Arai and Y. Matsuzawa, Lett. Math. Phys. **83**, 201 (2008).

- [16] M. Arndt, O. Nairz, J. Vos-Andreae, C. Keller, G. van der Zouw, and A. Zeilinger, *Nature* **401**, 680 (1999).
- [17] A. D. Baute, I. L. Egusquiza, J. G. Muga, and R. Sala Mayato, *Phys. Rev. A* **61**, 052111 (2000).
- [18] C. M. Bender, *Rep. Prog. Phys.* **70**, 947 (2007).
- [19] N. Bohr, *Nature* **121**, 580 (1928).
- [20] N. Bohr, in *Albert Einstein: Philosopher-Scientist*, edited by P. A. Schilpp (Cambridge Univ. Press, Cambridge, 1949). pp.199-242.
- [21] G. Bonneau, J. Faraut and G. Valent, *Am. J. Phys.* **69**, 322 (2001).
- [22] V. B. Braginsky and F. Y. Khalili, *Quantum Measurement* (Cambridge University Press, Cambridge, 1992).
- [23] N. Brunner, A. Acín, D. Collins, N. Gisin, and V. Scarani, *Phys. Rev. Lett.* **91**, 180402 (2003).
- [24] M. Brune, J. Bernu, C. Guerlin, S. Deleglise, C. Sayrin, S. Gleyzes, S. Kuhr, I. Dotsenko, J. M. Raimond, and S. Haroche, arXiv:0809.1511.
- [25] P. Busch, P. Mittelstaedt, and P. J. Lahti, *Quantum Theory of Measurement* (Springer-Verlag, Berlin, 1991).
- [26] K. M. Case, *Phys. Rev.* **80**, 797 (1950).
- [27] C. Caves, *Rev. Mod. Phys.* **52**, 341 (1980).
- [28] C. Caves, *Phys. Rev. Lett.* **54**, 2465 (1985).
- [29] L. Childress, M. V. Gurudev Dutt, J. M. Taylor, A. S. Zibrov, F. Jelezko, J. Wrachtrup, P. R. Hemmer, and M. D. Lukin, *Science* **314**, 281 (2006).
- [30] T. E. Clark, R. Menioff and D. H. Sharp, *Phys. Rev. D* **22**, 3012 (1980).
- [31] G. M. D'Ariano, to appear in *Philosophy of Quantum Information and Entanglement* edited by A. Bokulich and G. Jaeger (Cambridge University Press, Cambridge) arXiv:0807.4383.
- [32] B. Davies, *Integral Transforms and their Applications* (Springer-Verlag, New York, 1978).
- [33] E. B. Davies and J. T. Lewis, *Commun. Math. Phys.* **17**, 239 (1970).

- [34] D. Deutsch, Proc. R. Soc. Lond. A **400**, 97 (1985).
- [35] D. Deutsch, talk in "UK-Workshop 2009 on Quantum Information" at British Embassy in Japan on 1.22.2009.
- [36] P. A. M. Dirac, Proc. Roy. Soc. A **113**, 621 (1926).
- [37] M. V. Gurudev Dutt, L. Childress, L. Jiang, E. Togan, J. Maze, F. Jelezko, A. S. Zibrov, P. R. Hemmer, and M. D. Lukin, Science **316**, 1312 (2007).
- [38] A. Einstein, Ann. Phys. (in Berlin) **360**, 241 (1918).
- [39] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
- [40] E. Farhi and S. Gutmann, Int. J. Mod. Phys. A **5**, 3029-3051 (1990).
- [41] R. P. Feynman, Int. J. of Theor. Phys. **21**, 467 (1982).
- [42] T. Fülöp, T. Cheon and I. Tsutsui, Phys. Rev. A **66**, 052102 (2002).
- [43] T. Fülöp, Ph.D. thesis, University of Tokyo, 2005.
- [44] T. Fülöp, arXiv:0708.0866.
- [45] A. Furuta, talk in "Recent Issues and Prospects on Quantum Theory" at KEK, on 3.22.2008.
- [46] C. F. Gauss, *Theoria Combinationis Observationum Erroribus Minimis Obnoxiae* (Apud Henricum Dieterich, Gottingae, 1823).
- [47] P.-L. Giscard, arXiv:0901.0333.
- [48] S. Gleyzes, S. Kuhr, C. Guerlin, J. Bernu, S. Deléglise, U. B. Hoff, M. Brune, J. M. Raimond, and S. Haroche, Nature **446**, 297 (2007).
- [49] A. N. Gordeyev and S. C. Chhajlany, J. Phys. A **30**, 6893 (1997).
- [50] M. J. W. Hall, Phys. Rev. A **64**, 052103 (2001).
- [51] M. J. W. Hall, Phys. Rev. A **69**, 052113 (2004).
- [52] R. Hanson *et al.*, Science **320**, 352 (2008).
- [53] S. Haroche and J.-M. Raimond, *Exploring the Quantum: Atoms, Cavities, and Photons* (Oxford Univ. Press, Oxford, 2006).
- [54] M. Hayashi and F. Sakaguchi, J. Phys. A **33**, 7793 (2000).

- [55] W. Heisenberg, *Z. Phys.* **43**, 172 (1927).
- [56] W. Heisenberg, *The Physical Principles of the Quantum Theory* (University of Chicago Press, Chicago, 1930; Dover, New York, 1949, 1967).
- [57] C. W. Helstrom, *Int. J. Theor. Phys.* **11**, 357 (1974).
- [58] P. A. Hiskett, D. Rosenberg, C. G. Peterson, R. J. Hughes, S. Nam, A. E. Lita, A. J. Miller, and J. E. Nordholt, *New J. Phys.* **8** 193 (2006).
- [59] F. Hiroshima, S. Kuribayashi, and Y. Matsuzawa, arXiv:0810.2350.
- [60] H. F. Hofmann, arXiv:0805.0029.
- [61] A. S. Holevo, *Rep. Math. Phys.* **13**, 379 (1978).
- [62] A. S. Holevo, *Rep. Math. Phys.* **16**, 385 (1979).
- [63] A. S. Holevo, *Probabilistic and statistical aspects of quantum theory* (North-Holland, Amsterdam, 1982).
- [64] A. S. Holevo, *Statistical Structure of Quantum Theory* (Springer, Berlin, 2001).
- [65] O. Hosten and P. Kwiat, *Science* **319**, 787 (2008).
- [66] M. Hotta and M. Ozawa, *Phys. Rev. A* **70**, 022327 (2004).
- [67] R. A. Hulse and J. H. Taylor, *Astrophysical Journal* **195**, L51 (1975).
- [68] C. J. Isham, *Lectures on Quantum Theory – Mathematical and Structural Foundations* – (Imperial College Press, London, 1995).
- [69] S. Ishii, *Heisenberg’s microscopy* (Nikkei Business Publication, Tokyo, 2006) (in Japanese).
- [70] S. Ishikawa, *Int. J. Theor. Phys.* **30**, 401 (1991).
- [71] S. Ishikawa, *Rep. Math. Phys.* **29**, 257 (1992).
- [72] K. Itô, *An Introduction to Probability Theory* (Cambridge University Press, Cambridge, 1978).
- [73] F. Jelezko *et al.*, *Phys. Rev. Lett.* **92**, 076401 (2004).
- [74] L. M. Johansen, *Phys. Lett. A* **322**, 298 (2004).
- [75] P. Jordan, *Z. Phys.* **37**, 383 (1926).

- [76] R. Jozsa, Phys. Rev. A **76**, 044103 (2007).
- [77] S. Kagami, Master Thesis at Tokyo Institute of Technology (2009).
- [78] E. H. Kennard, Z. Phys. **44**, 326 (1927).
- [79] N. Konno, Quantum Inf. Process. **1**, 345 (2002).
- [80] N. Konno, J. Math. Soc. Jpn. **57**, 1179 (2005).
- [81] N. Konno, *Mathematics on Quantum Walk* (Sangyo Tosho, Tokyo, 2008) (in Japanese).
- [82] K. Koshino and A. Shimizu, Phys. Rep. **412** 191 (2005).
- [83] A. M. Krall, J. Diff. Eq. **45**, 128 (1982).
- [84] K. Kraus, Ann. Phys. **64**, 311 (1971).
- [85] L. D. Landau and E. M. Lifschitz, *Quantum Mechanics Non-Relativistic Theory* (Pergamon Press, 1977) Third edition.
- [86] U. Larsen, J. Phys. A **23**, 1041 (1990).
- [87] J. Maddox, Nature **331**, 559 (1988).
- [88] L. Mandelstam and I. Tamm, J. Phys.(USSR) **9**, 249 (1945).
- [89] N. Margolus and L. B. Levitin, Physica D **120**, 188 (1998).
- [90] K. Mølmer, Phys. Lett. A **292**, 151 (2001).
- [91] G. Mitchison, R. Jozsa, and S. Popescu, Phys. Rev. A **76**, 062105 (2007).
- [92] J. von Neumann, Math. Ann. **102**, 49 (1929).
- [93] J. von Neumann, *Mathematische Grundlagen der Quantumechanik* (Springer, Berlin, 1932), [Eng. trans. by R. T. Beyer, *Mathematical foundations of quantum mechanics* (Princeton University Press, Princeton, 1955).]
- [94] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
- [95] I. Ojima, in Proceeding of " *Quantum Bio-Informatics – From Quantum Information to Bio-Information –* " at Tokyo University of Science on 3.14-17.2007. pp. 217–228. arXiv:0705.2945.

- [96] M. Onoda, S. Murakami, and N. Nagaosa, **93**, 083901 (2004).
- [97] M. Ozawa, J. Math. Phys. **25**, 79-87 (1984).
- [98] M. Ozawa, Phys. Rev. Lett. **60**, 385 (1988).
- [99] M. Ozawa, in *Squeezed and Nonclassical Light*, edited by P. Tombesi and E. R. Pike, (Plenum, New York, 1989), pp. 263- 286.
- [100] M. Ozawa, in *Quantum Aspects of Optical Communications* (Springer, Heidelberg, 1991) pp. 1-17.
- [101] M. Ozawa, Phys. Rev. A **67**, 042105 (2003).
- [102] M. Ozawa, J. Quant. Inf. **1**, 569 (2003).
- [103] M. Ozawa, Ann. Phys. **311**, 350-416 (2004).
- [104] M. Ozawa, J. Opt. B Quantum Semiclass. Opt. **7**, S672 (2005).
- [105] A. Pais, ”*Subtle is the Lord... The Science and Life of Albert Einstein*” (Oxford University Press, Oxford, 1982).
- [106] A. D. Parks, D. W. Cullin, and D. C. Stoudt, Proc. R. Soc. Lond. A **454**, 2997 (1998).
- [107] A. D. Parks, J. Phys. A **41**, 335305 (2008).
- [108] L. Pauling and E. B. Wilson, *Introduction to Quantum Mechanics* (Mcgraw-Hill, New York, 1935).
- [109] A. Peres and P. Scudo, J. Mod. Opt. **49**, 1235 (2002).
- [110] G. J. Pryde, J. L. O’Brien, A. G. White, T. C. Ralph, and H. M. Wiseman, Phys. Rev. Lett. **94**, 220405 (2005).
- [111] M. Rédei and S. J. Summers, Stud. Hist. Philos. Sci. B **38**, 390 (2007).
- [112] M. Reed and B. Simon, *Methods of Modern Mathematical Physics II, Fourier Analysis, Self-Adjointness* (Academic Press, New York, 1975).
- [113] F. Rellich, Math. Ann. **122**, 343-368 (1950).
- [114] K. J. Resch, J. S. Lundeen, and A. M. Steinberg, Phys. Lett. A **324**, 125 (2004).
- [115] K. J. Resch, Science **319**, 733 (2008).

- [116] B. Reznik and Y. Aharonov, *Phys. Rev. A* **52**, 2538 (1995).
- [117] N. W. M. Ritchie, J. G. Story, and R. G. Hulet, *Phys. Rev. Lett.* **66**, 1107 (1991).
- [118] H. P. Robertson, *Phys. Rev.* **34**, 163-164 (1929).
- [119] D. Rohrlich and Y. Aharonov, *Phys. Rev. A* **66**, 042102 (2002).
- [120] R. A. Sack, *Phil. Mag.* **3**, 497 (1958).
- [121] T. Sagawa, Master Thesis, Tokyo Institute of Technology, 2008.
- [122] H. Salecker and E. P. Wigner, *Phys. Rev.* **109**, 571 (1958).
- [123] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, 1965) Third edition.
- [124] E. Schrödinger, *Sitz. Ber. Preuss. Akad. Wiss., Phys.-Math. Kl.* (Berlin), 296 (1930).
- [125] A. Shapere and F. Wilczek, *Geometric Phases in Physics* (World Scientific, Singapore, 1988).
- [126] Y. Shikano and A. Hosoya, *J. Math. Phys.* **49**, 052104 (2008).
- [127] Y. Shikano and A. Hosoya, arXiv:0812.4502.
- [128] E. Sjöqvist, *Phys. Lett. A* **359**, 187 (2006).
- [129] E. Sjöqvist, *Acta Phys. Hung. B* **26**, 195 (2006).
- [130] A. M. Steinberg, *Phys. Rev. Lett.* **74**, 2405 (1995).
- [131] A. Tanaka, *Phys. Lett. A* **297** 307 (2002).
- [132] A. Tonomura, J. Endo, T. Matsuda, T. Kawasaki, and H. Ezawa, *Am. J. Phys.* **57**, 117 (1989).
- [133] I. Tsutsui, T. Fülöp and T. Cheon, *J. Phys. A* **36**, 275-287 (2003).
- [134] J. Twamley and G. J. Milburn, *New J. of Phys.* **8**, 328 (2006).
- [135] A. Uhlmann, *Rep. Math. Phys.* **24**, 229 (1986).
- [136] A. Uhlmann, *Lett. Math. Phys.* **21**, 229 (1991).
- [137] A. Uhlmann, in *Proceedings of the First Max Born Symposium*, edited by R. Giel-erak, J. Lukierski, and Z. Popowicz (Kluwer Academics Press, Netherlands, 1992) pp. 267-274. mp-arc:92-44.

- [138] L. Vaidman, Am. J. Phys. **60**, 183 (1992).
- [139] L. Vaidman, Found. Phys. **26**, 895 (1996).
- [140] M. S. Wang, Phys. Rev. Lett. **79**, 3319 (1997).
- [141] H. Weyl, Math. Ann. **68**, 220-269 (1910).
- [142] H. Weyl, *Gryppentheorie und Quantenmechanik* (S. Hirzel, Leipzig, 1928), [Eng. Trans. H. P. Robertson, *The Theory of Groups and Quantum Mechanics* (Dover, New York, 1950).].
- [143] N. S. Williams and A. N. Jordan, Phys. Rev. Lett. **100**, 026804 (2008).
- [144] H. M. Wiseman, Phys. Rev. A **65**, 032111 (2002).
- [145] H. P. Yuen, Phys. Rev. Lett. **51**, 719 (1983).
- [146] K. Yokota, T. Yamamoto, M. Koashi, and N. Imoto, arXiv:0811.1625.
- [147] ”*The Quantum and the Foundations of Physics Conference*” at the University of Texas organized by J. A. Wheeler on 12.17-20.1982.