On a Quantum Walk

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Welcome to your questions in Japanese or English.

Open the “unopened” door

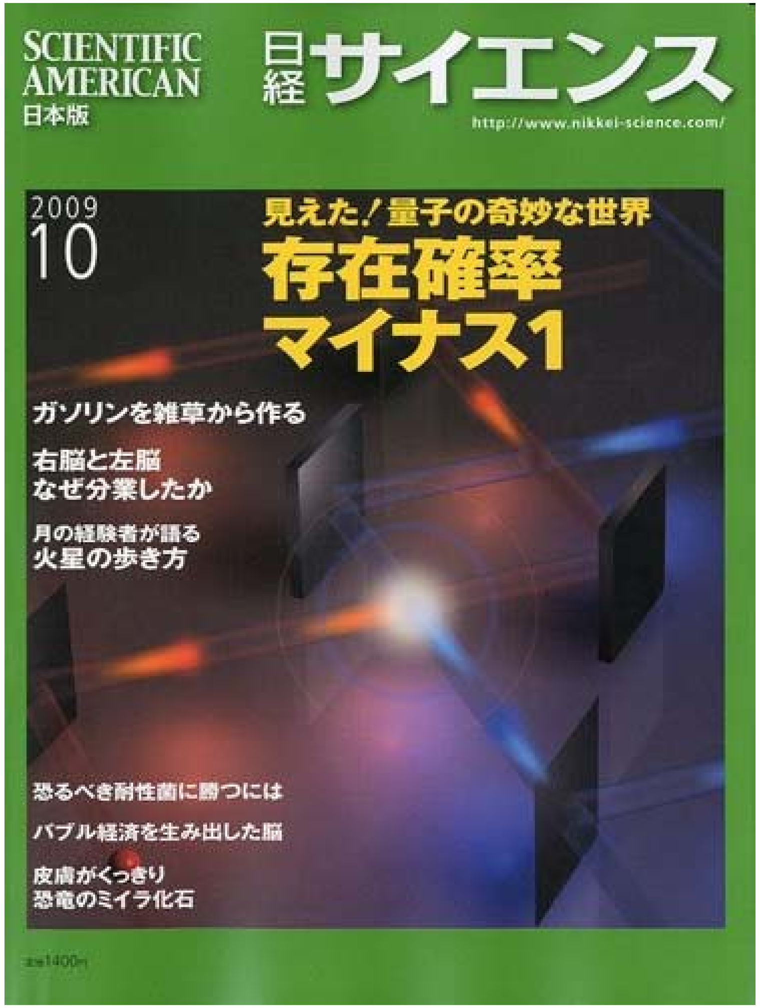
青不動王 (Blue Fudo-Oh)

@ 青蓮院門跡 (Syoreiin-Monseki)

This is first opened at 青蓮院門跡 since the beginning.

Location: Around 知恩院(Chiouin)
30 min Walk with the south direction along 東大路通(East-Ohji Street) the from here.

Open the “unopened” door in Quanta.



1. Last year, I talked about the weak value in this conference.

(日経サイエンス 2009年10月号, 解説記事 / Yakir Aharonovインタビュー監修)

2. Private Quantum Internet (秘密を守る量子グーグル) by Seth Lloyd
(日経サイエンス 2010年1月号, 2009年11月25日発売予定, 翻訳監修)

Today’s Aim

What “unopened” doors are potential to open by a quantum walk?

Collaborators

Outline

1. What is Quantum Walk?
2. Weak Limit Theorem
3. Mathematical Result on the Decoherence Quantum Walk
    1. Randomness
    2. Application: Quantum Measurement Problem
4. Conclusion and Open Questions

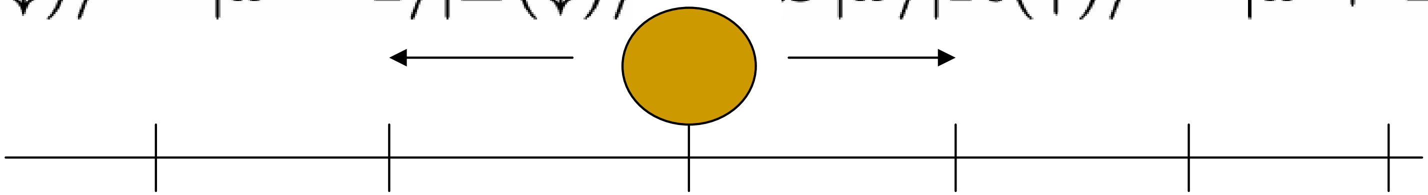
Random Walk v.s. Quantum Walk

One of the motivations to define the quantum walk was the quantum analogue of the random walk.

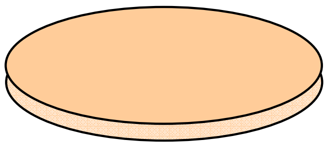
(Y. Aharonov, L. Davidovich, and N. Zagury, Phys. Rev. A **48**, 1687 (1993))

Shift: due to the coin state

$$S|x\rangle|L(\downarrow)\rangle = |x-1\rangle|L(\downarrow)\rangle \qquad S|x\rangle|R(\uparrow)\rangle = |x+1\rangle|R(\uparrow)\rangle$$



Walker Space $\mathcal{L}^2(\mathbb{Z})$



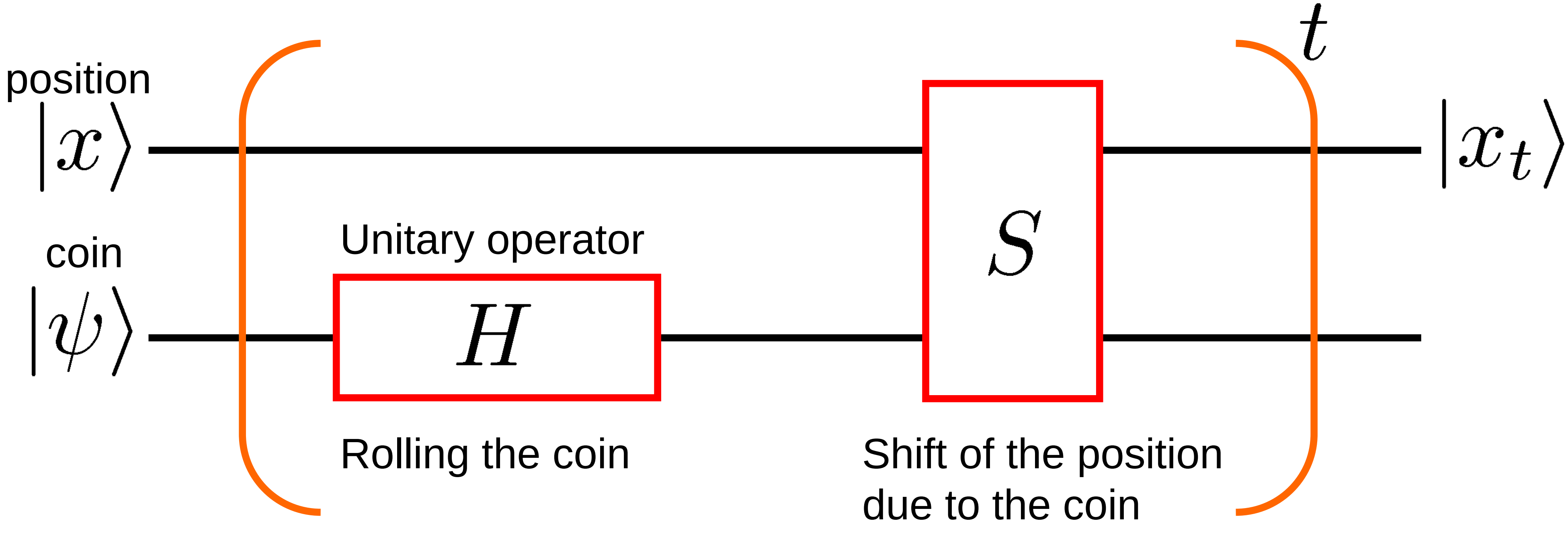
Coin Flip: Dynamics of Quantum System, Unitary Operator

Coin Space: independent of Walker Space

$\mathbb{C}^2$ Qubit

$$H = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$$

Quantum Circuit Representation



Classical Walk

$$x, x_t \in \mathbb{Z}$$

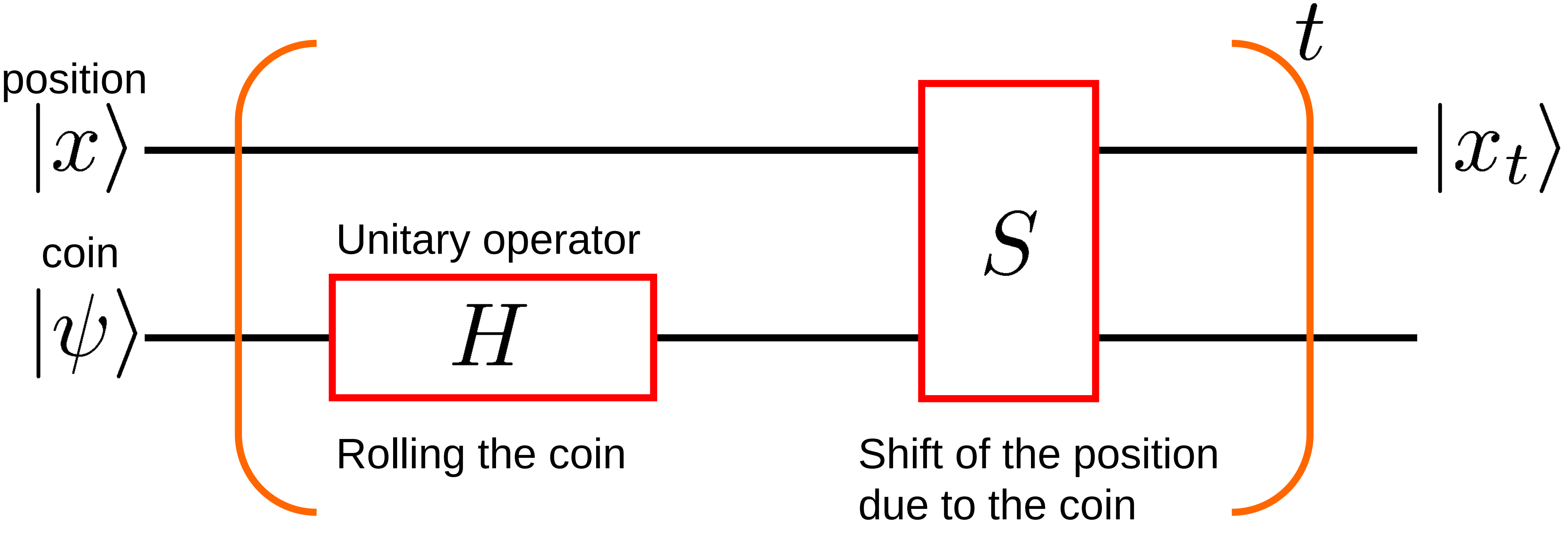
$$\psi \in \{\uparrow, \downarrow\}$$

Quantum Walk

$$x, x_t \in \mathbb{Z}$$

$$|\psi\rangle \in \mathcal{H}_2$$

Probability Distribution



Classical Walk

$$\Pr(x_t) = \sum_{i+j=x_t+x} p^i (1-p)^j$$

Quantum Walk

$$\Pr(x_t) = \left| \langle x_t | \, \mathrm{Tr}_c (SH)^t |x\rangle\langle x| \otimes |\psi\rangle\langle\psi| ((SH)^\dagger)^t |x_t\rangle \right|^2$$

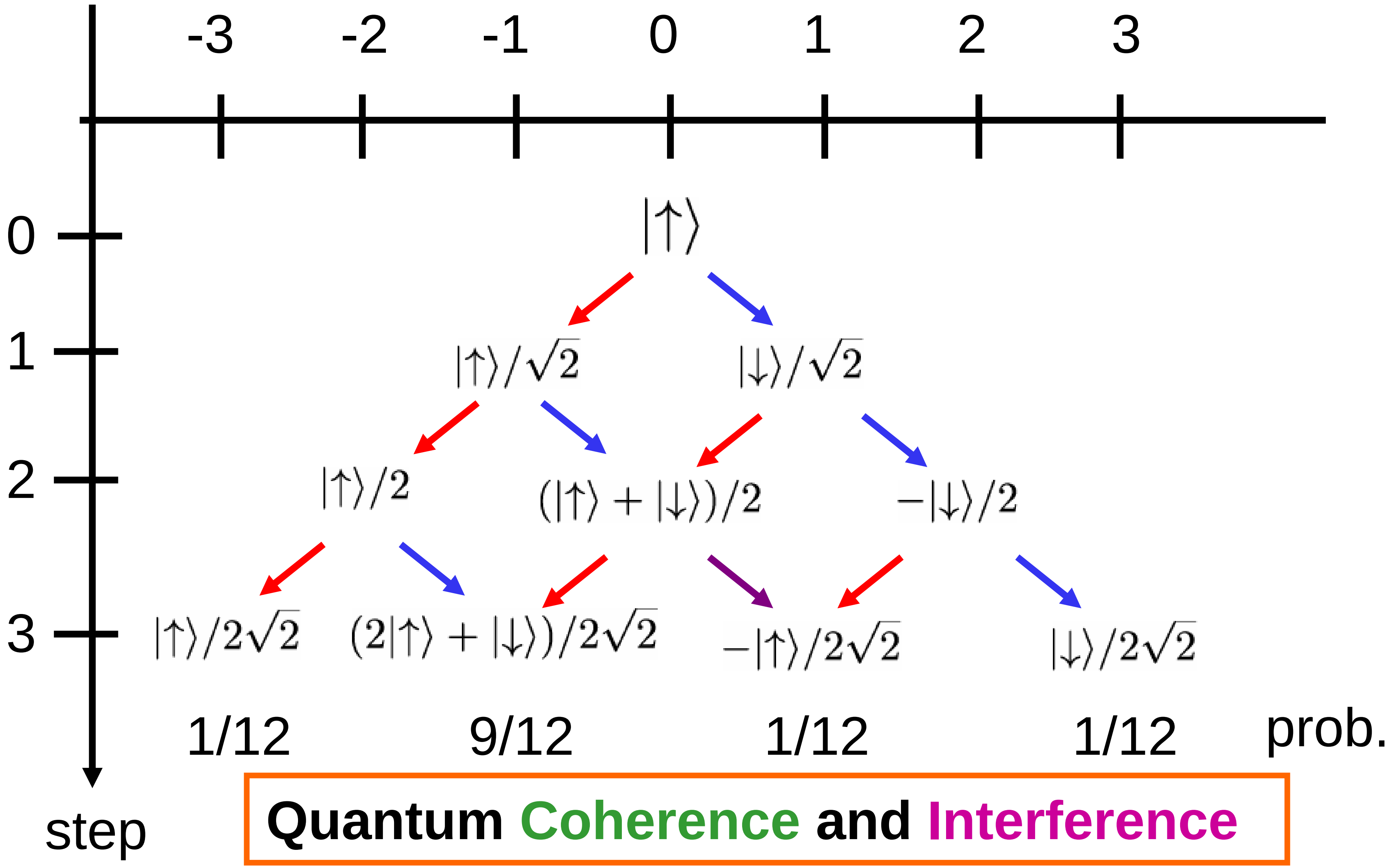
Example of Quantum Walk

- Initial Condition
  - Position: x = 0 (localized)
  - Coin: $|\uparrow\rangle$
- Coin Operator: Hadamard Coin

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

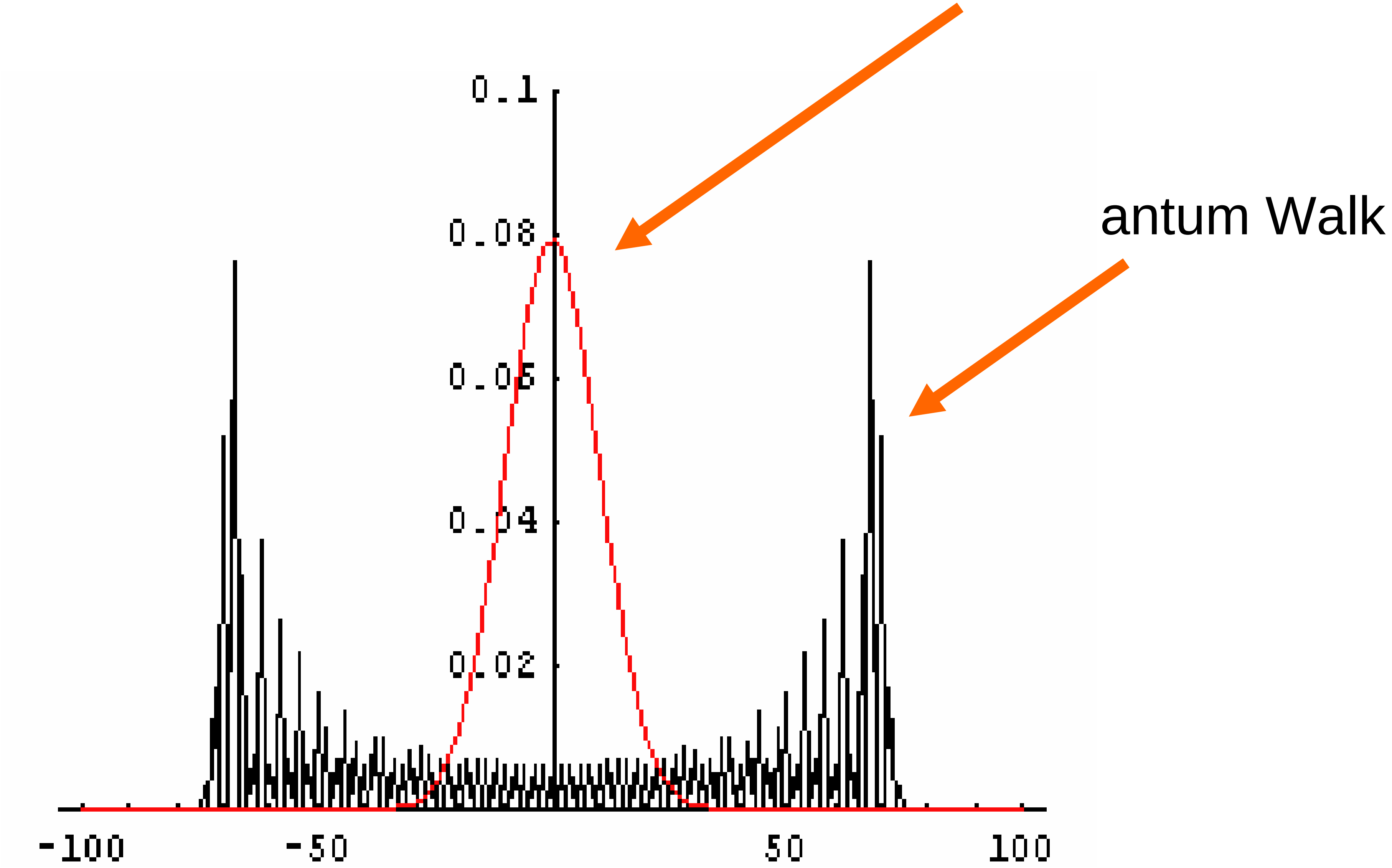
Let's see the dynamics of quantum walk by 3 step!!

Example of Quantum Walk



Quantum Coherence and Interference

100 step in Walks



Weak Limit Theorem

(Convergence in terms of the distribution)

Random Walk

Central Limit Theorem

$$\frac{X_t}{\sqrt{t}} \Rightarrow N(0,1)$$

in the case of right and left probability ½.

Quantum Walk

N. Konno, Quantum Information Processing **1**, 345 (2002)

$$\frac{X_t}{t} \Rightarrow K(|a|)$$

with the initial coin state

$$\rho_0 = (|L\rangle\langle L| + |R\rangle\langle R|)/2$$

density

$$\mu_K(x;r) = \frac{\sqrt{1-r^2}}{\pi(1-x^2)\sqrt{r^2-x^2}} I_{(-r,r)}(x)$$

Experimental Realizations and Applications

- Experimental Realizations
  - Trapped Atom with Optical Lattice and Ion Trap
    - M. Karski *et al.* Science 325, 174 (2009). 23 step
    - C. Roos, in private communication. 15 step?
  - Photon in Linear Optics (restricted)
    - A. Schreiber *et al.* arXiv:0910.2197. 5 step
- Applications
  - Universal Quantum Computation
    - A. Childs, Phys. Rev. Lett. **102**, 180501 (2009).
    - N. B. Lovett *et al.* arXiv:0910.1024.

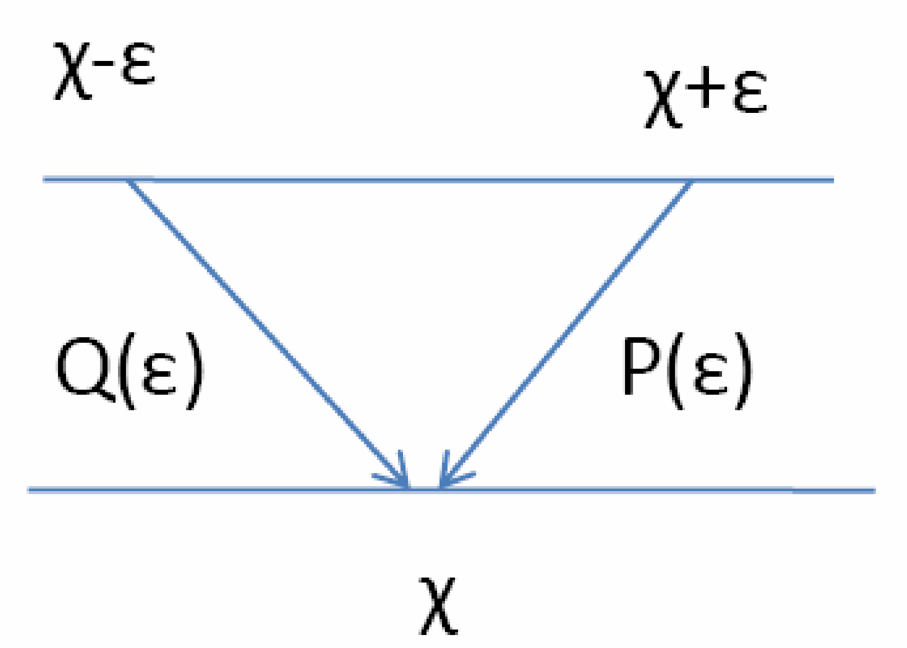
Realization of Quantum Walk = Quantum Computational Device

Dirac Equation and Quantum Walk

Quantum Coin Flip

$$\widetilde{H}^{(S)}(\epsilon) = \begin{bmatrix} \cos(\epsilon) & -i\sin(\epsilon) \\ -i\sin(\epsilon) & \cos(\epsilon) \end{bmatrix}$$

Note that this cannot represents arbitrary coin flip.



Time Evolution of Quantum Walk

$$\Psi_{n+1}(\chi) = Q(\epsilon)\Psi_n(\chi-\epsilon) + P(\epsilon)\Psi_n(\chi+\epsilon)$$

$$Q(\epsilon) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - i\epsilon \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + O(\epsilon^2) \quad P(\epsilon) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - i\epsilon \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + O(\epsilon^2)$$

Dirac Equation from Quantum Walk

$$\Psi_\tau(x) = \left(I - i\epsilon(\sigma_X + \sigma_Z \partial_x) + O(\epsilon^2)\right)^\tau \Psi_0(x)$$

$$\sim e^{-i(\sigma_X + \sigma_Z \partial_x)t} \Psi_0(x)$$

$$t = \epsilon\tau$$

$$\boxed{i\frac{\partial \Psi_t(x)}{\partial t} = \left(i\frac{\partial}{\partial x}\sigma_Z + \sigma_X\right)\Psi_t(x)}$$

Motion of Dirac Particle : Walker Space

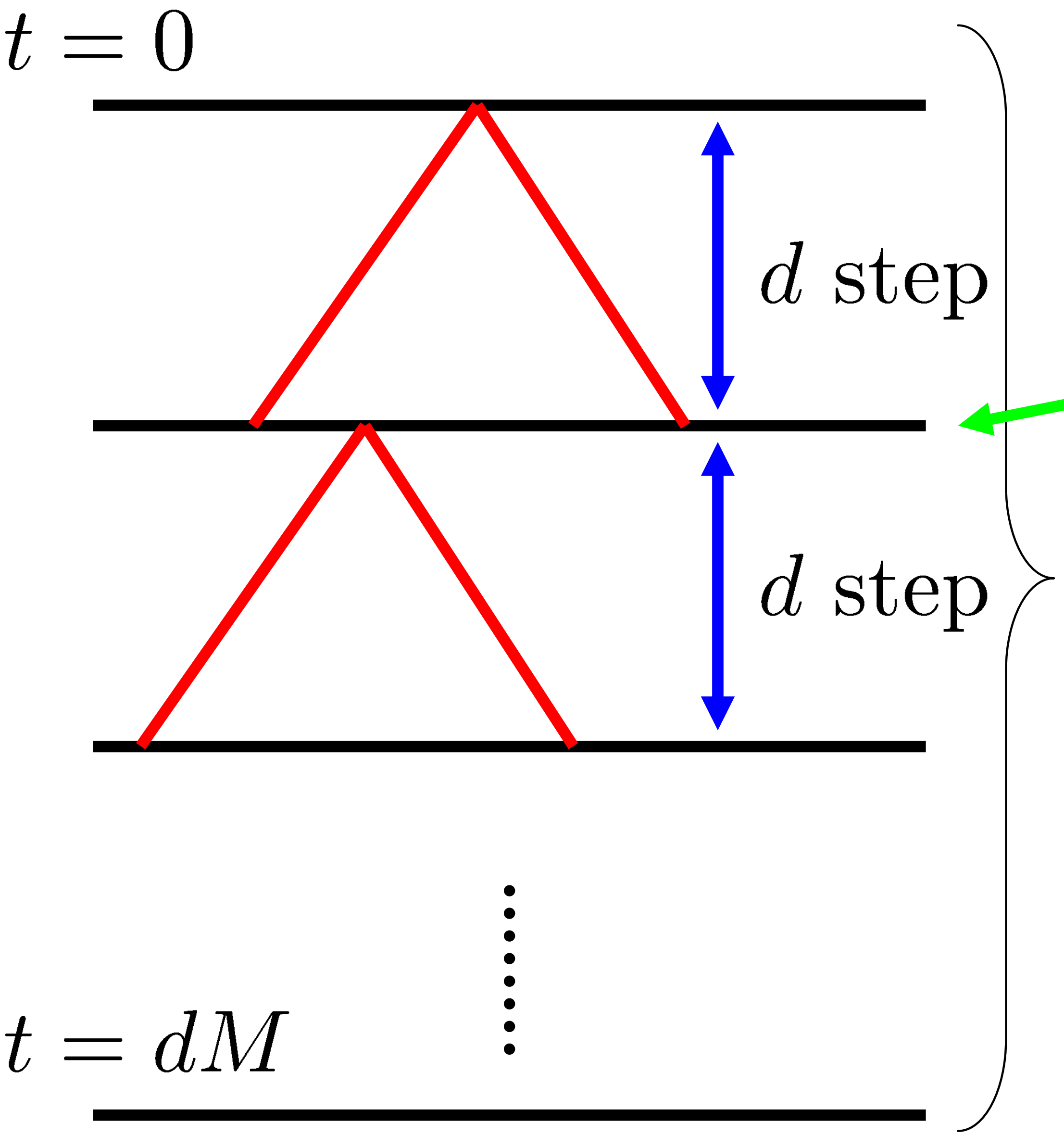
Spinor : Coin Space

Summary of Quantum Walk

- Definition
  - Quantum Analogue of the Random Walk
- Important Properties from the coherence and interference
  - Deterministic Dynamics
  - Faster Diffusion like the Ballistic Transport
  - Inverted-bell Shaped Limit Distribution
- Any quantum walk can be realized by the artificial control.
- Some quantum walk can be found in nature.

Decoherence Quantum Walk

K. Zhang, Ph. D thesis at Temple University 2008

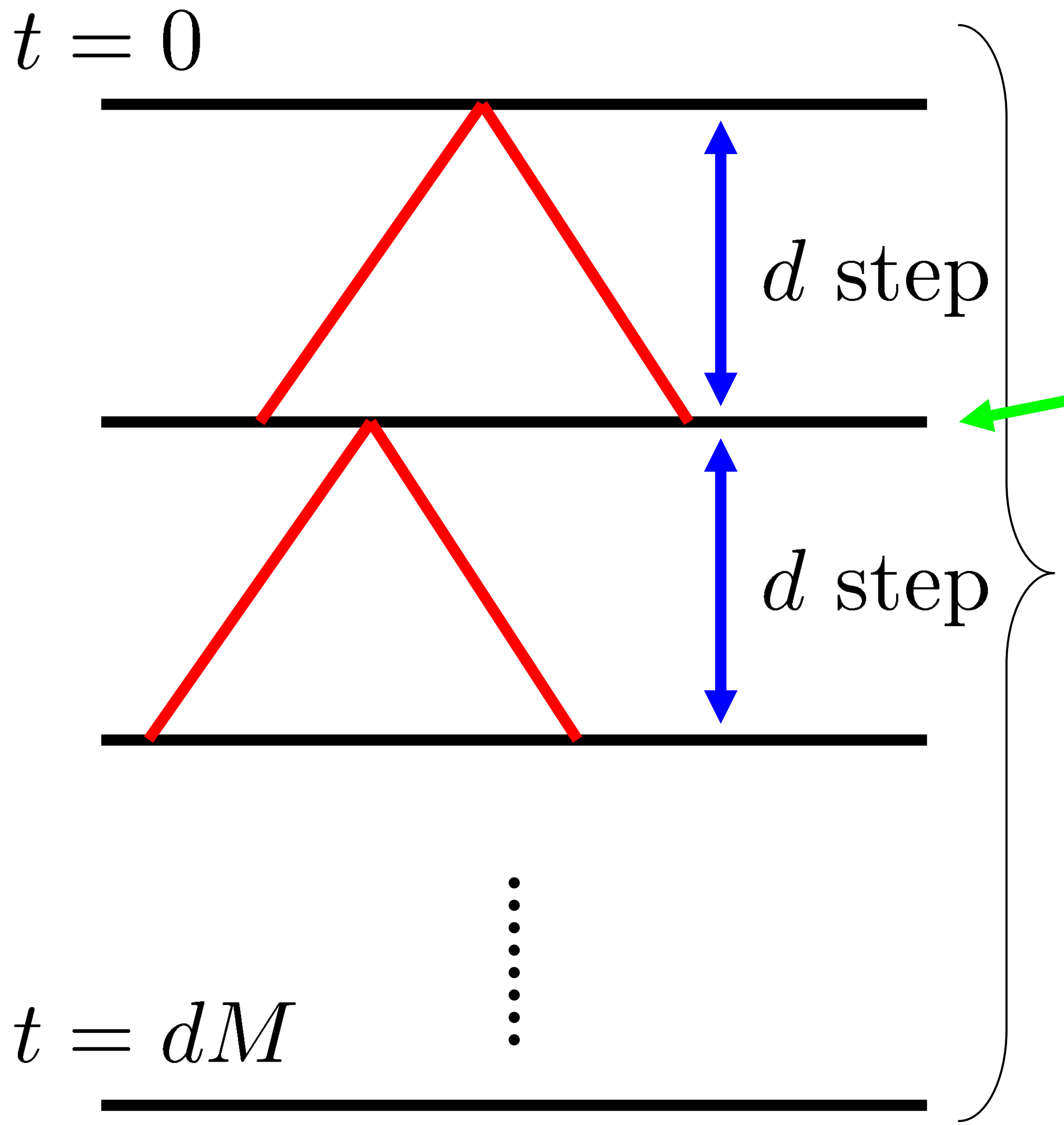


To measure the position and the coin.

M:
Number of Measurement

$$\frac{X_t}{t^{1/2}} \Rightarrow N(\mu, \nu)$$

Simplified Model



$$\rho_0 = (|L\rangle\langle L| + |R\rangle\langle R|)/2$$

To measure the position and the coin and prepare the new coin.

M:
Number of Measurement

Main Theorem

Let $\{Y_i^{(d)}\}$ be a sequence of an i.i.d. SQW on $\mathbb{Z}$ with the initial localized position $x = 0$, the initial coin qubit $\rho_0 = (|L\rangle\langle L| + |R\rangle\langle R|)/2$, and the quantum coin flip $H = a|L\rangle\langle L| + b|L\rangle\langle R| + c|R\rangle\langle L| + d|R\rangle\langle R| \in SU(2)$ noting that $abcd \neq 0$. Let $X_t = \sum_{i=1}^{M} Y_i^{(d)}$ be a random variable on a position with the time period d between measurements and the number of the measurements M with $t = dM$. If $\underline{d \sim t^\beta}$, then, as $t \to \infty$, we obtain the limit distribution as

$$\boxed{\frac{X_t^{(\beta)}}{t^{(1+\beta)/2}} \Rightarrow \begin{cases} N(0,1) & \text{on } \beta = 0 \\ N(0, 1 - \sqrt{1 - |a|^2}) & \text{on } 0 < \beta < 1 \\ K(|a|) & \text{on } \beta = 1, \end{cases}} \tag{1}$$

Our Result

Changing the measurement timing.

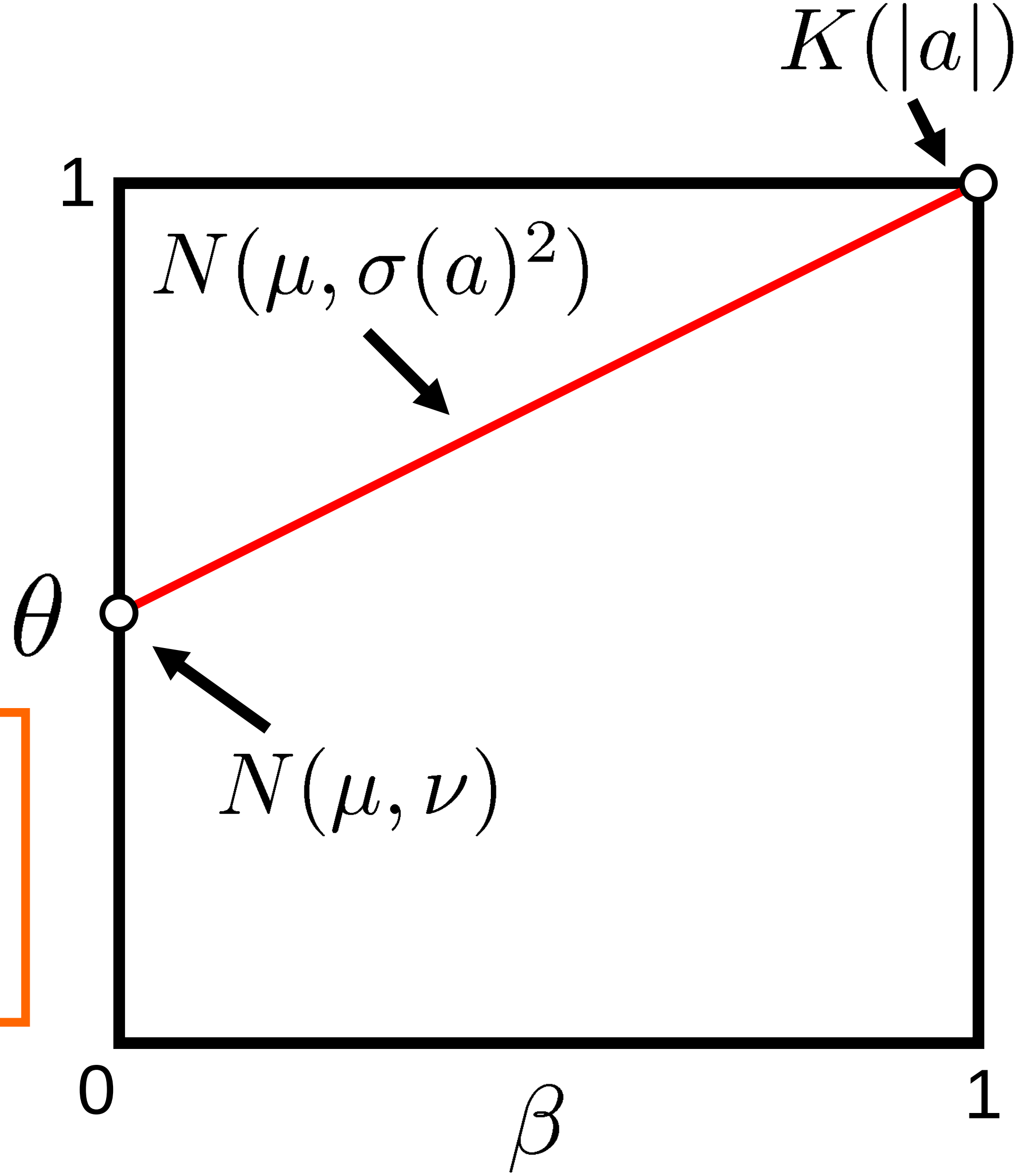
$$t = dM$$

$$d \sim t^{\beta}$$

$M \sim t^{1-\beta}$: Number of Measurement

After the measurement for the position and the coin, the position depends on the measurement result and the coin state is reset.

θ: Time Scaling Order



Proof of Our Result

Since $\{Y_i^{(d)}\}$ is i.i.d. sequence and $X_t = \sum_{i=1}^{M} Y_i^{(d)}$, we can describe the characteristic function $E(e^{i\xi X_t})$ as

$$E(e^{i\xi X_t}) = \left\{E(e^{i\xi Y_1^{(d)}})\right\}^M = \left\{\int_0^{2\pi} \frac{1}{2}\mathrm{Tr}[\widehat{H}^d(k+\xi)\cdot\widehat{H}^{-d}(k)]\frac{dk}{2\pi}\right\}^M, \quad (1)$$

where $\widehat{H}(k) = (e^{ik}|R\rangle\langle R| + e^{-ik}|L\rangle\langle L|)H$. Let an eigenvalue of $\widehat{H}(k)$ be $e^{i\varphi(k)}$ and $h(k) \equiv d\varphi(k)/dk$. Then, we can rewrite the integrand of Eq. 1 for replacing ξ with ξ/t^θ and d with t^β in the following : assuming $\theta > \beta$, then we have

$$\frac{1}{2}\mathrm{Tr}[\widehat{H}^{t^\beta}(k+\xi/t^\theta)\cdot\widehat{H}^{-t^\beta}(k)] = 1 - \frac{\xi^2}{2t^{2(\theta-\beta)}}(h(k))^2 + o(t^{-2(\theta-\beta)}), \quad (2)$$

where $o(\cdot)$ is the small o notation. By Eqs. 1 and 2, only if $2(\theta-\beta) = 1-\beta$, then there exists the limit distribution as follows:

$$E(e^{i\xi X_t^{(\beta)}/t^\theta}) \to e^{-\xi^2\sigma(a)^2/2} \text{ as } t\to\infty, \quad (3)$$

where

$$\sigma(a)^2 = \int_0^{2\pi}(h(k))^2\frac{dk}{2\pi} = 1-\sqrt{1-|a|^2}. \quad (4)$$

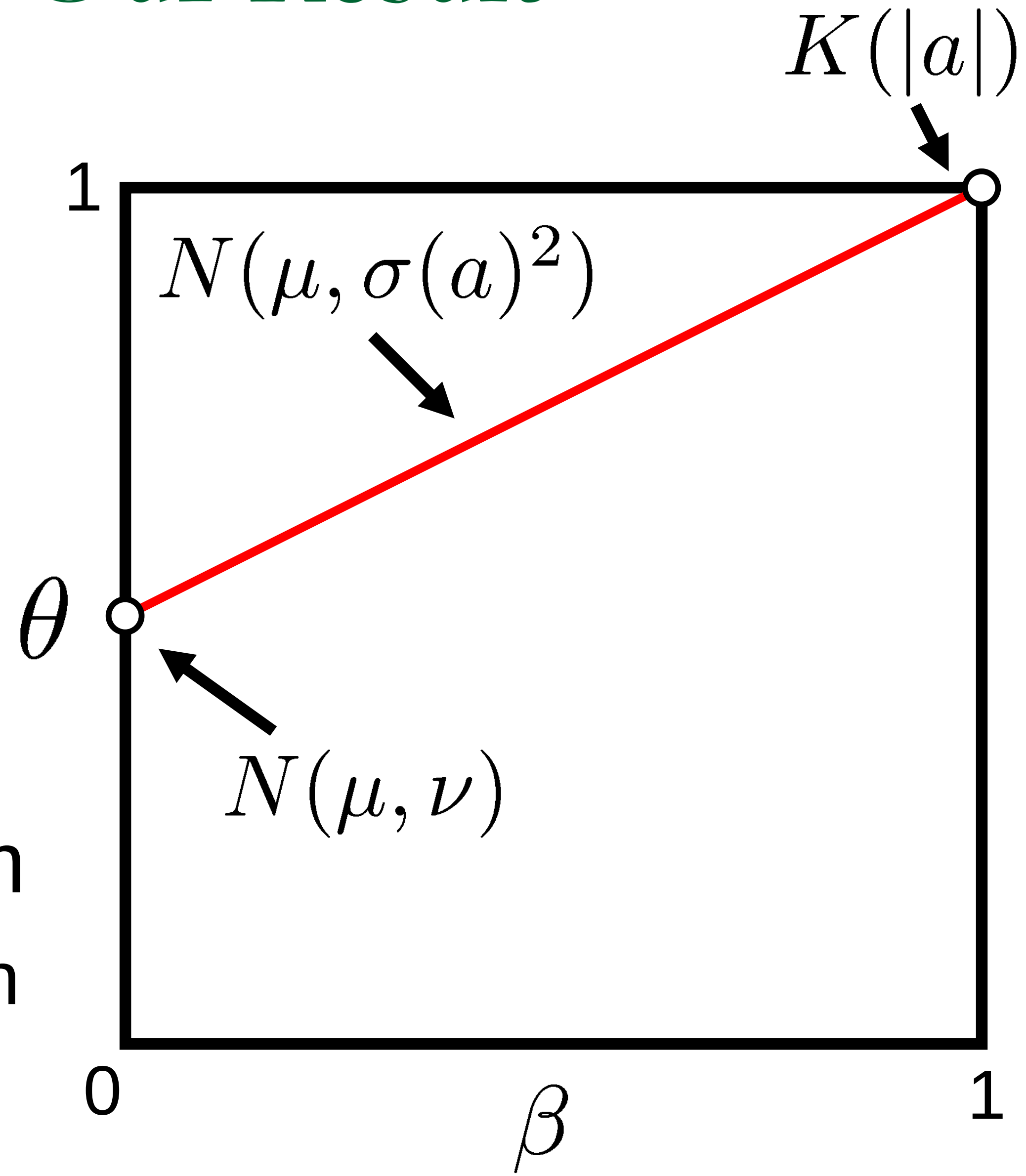
Spatial Fourier Transform

Eigenvalue Analysis

Taylor Expansion

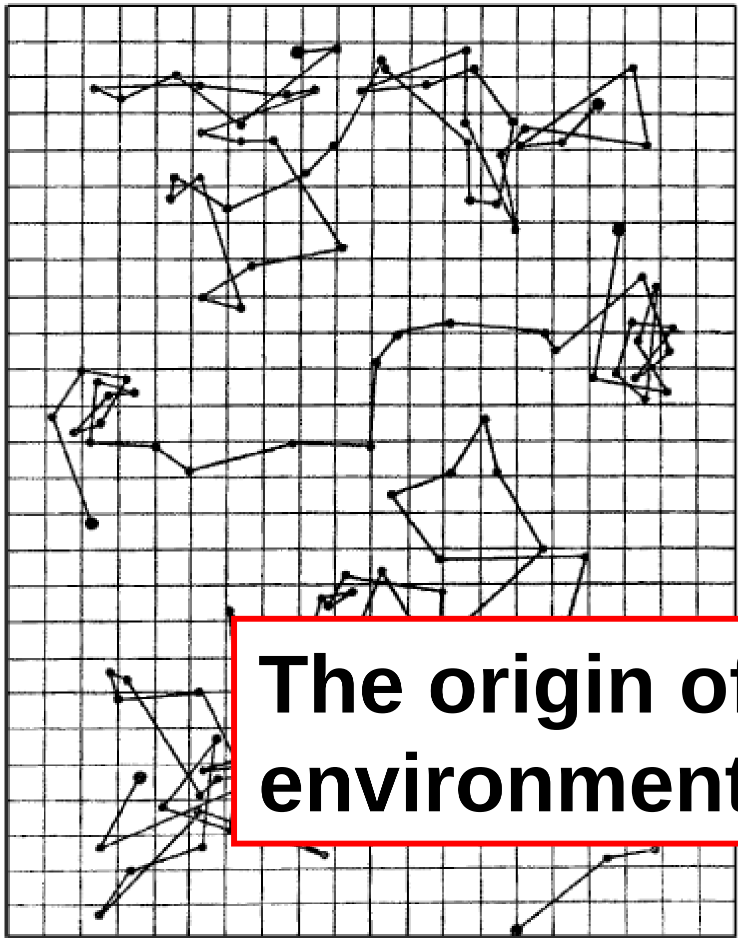
Interpretations for Our Result

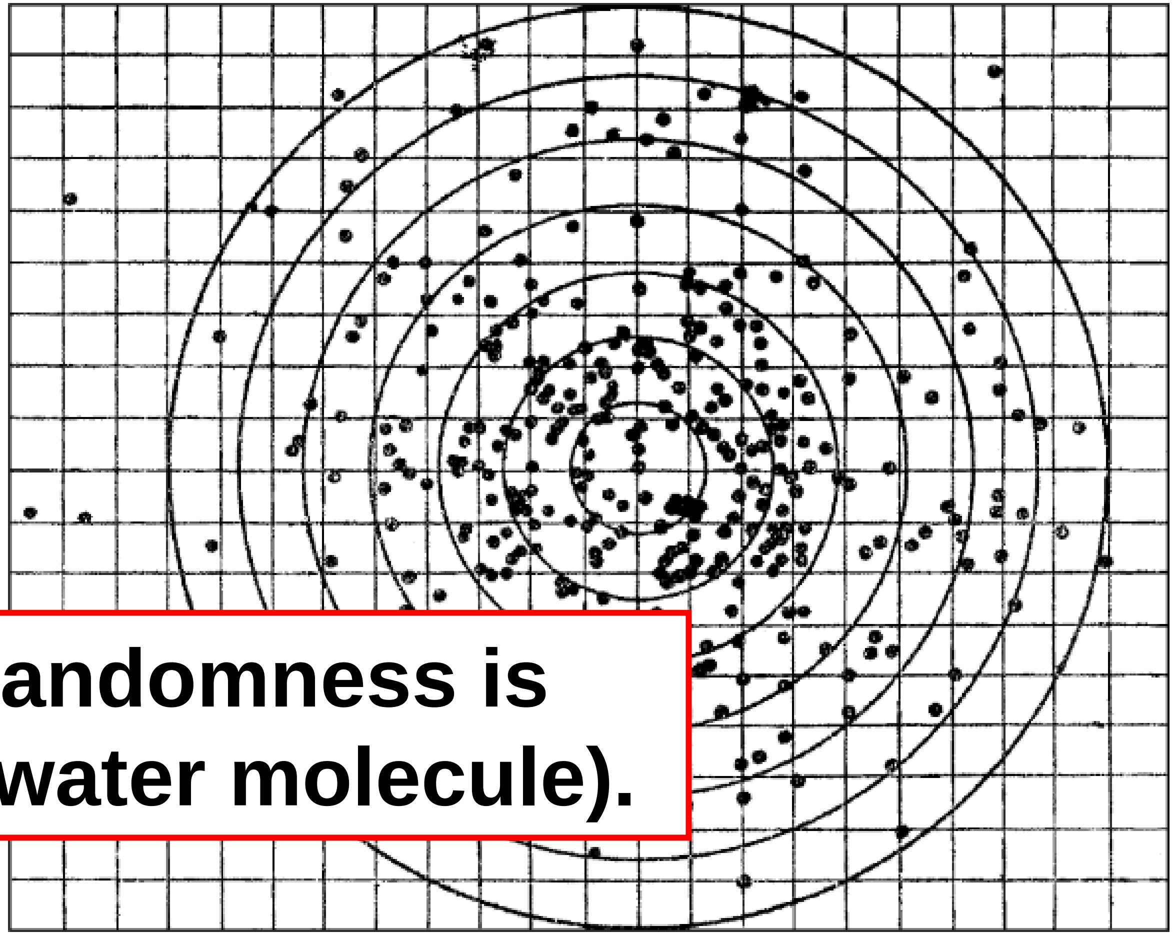
- Randomness
  - Where does it come from?
  - Can we define the quantity of the randomness?
- Measurement Problem
  - Based on "Copenhagen Interpretation"



Demonstration of Random Walk

Brownian motion was demonstrated by Perrin using the limit distribution of the Pollen.





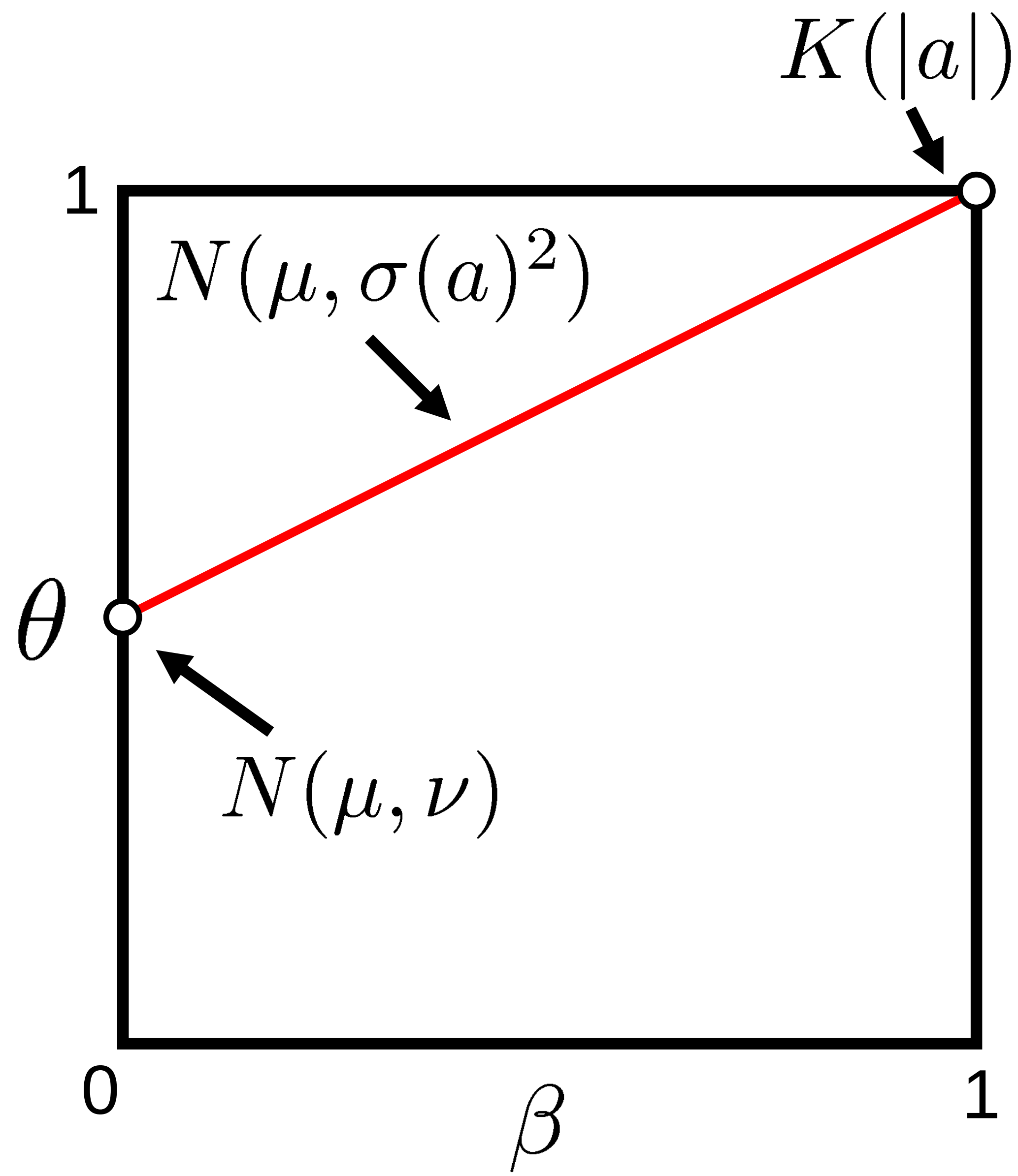
The origin of randomness is environment (water molecule).

J. Perrin, Ann. chim. et d. phys. VIII **18**, 5 (1909).

Randomness in Quantum Walk

- The description of environment corresponds to the one of quantum measurement.

- Quantification of randomness
  - There does not exist quantum-classical transition.



Application: Quantum Measurement Problem

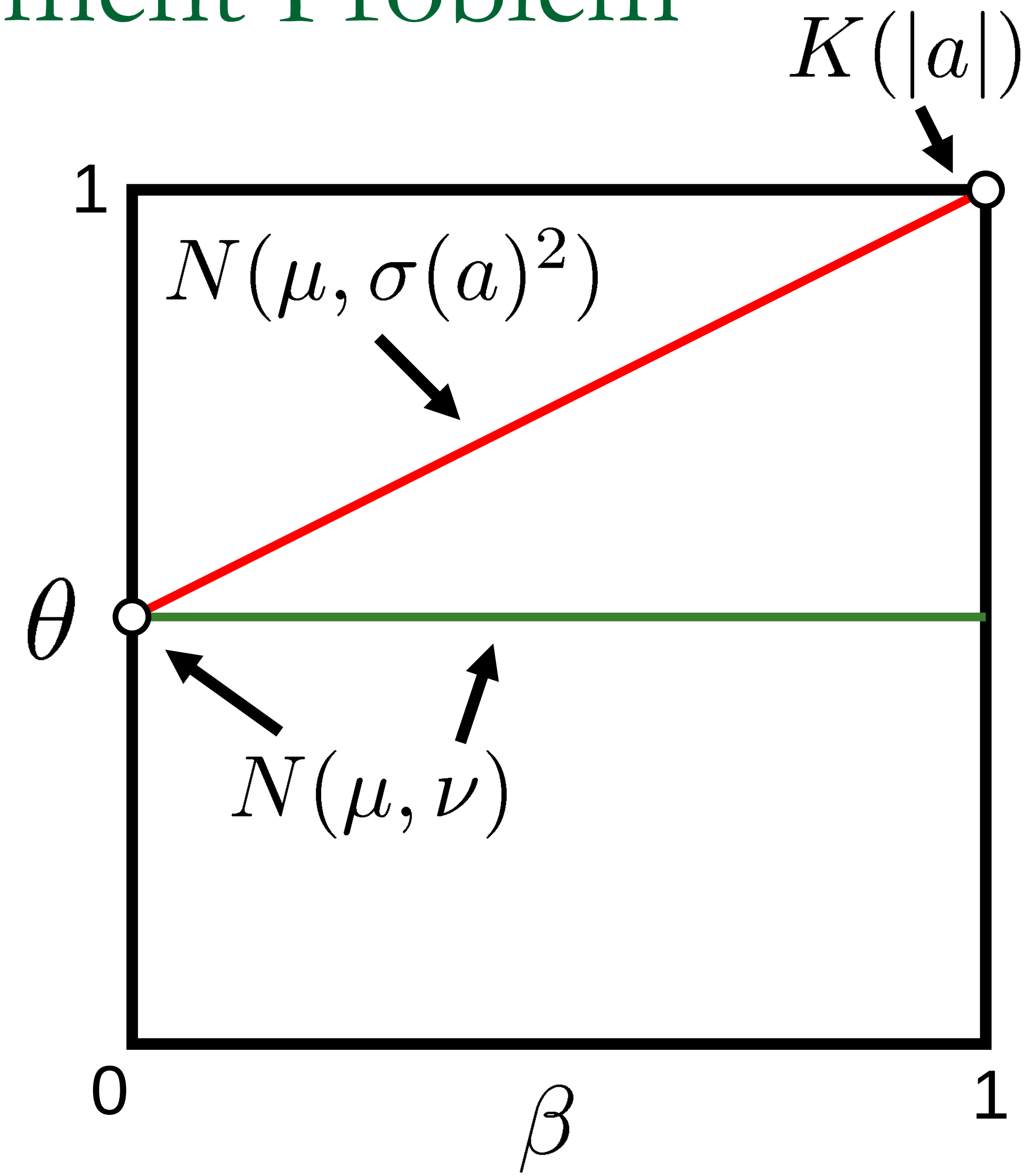
- Based on "Copenhagen Interpretation"
  1. We cannot know its dynamics.
  2. We can know its dynamics.
- Projective Measurement
  - To forget the memory
  - Quantum-classical transition, state reduction

Conjecture (H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems*)

The dynamics of the projective measurement can be taken as the Markov process.

Quantum Measurement Problem

- If the projective measurement can be taken as the Markov process (Conjecture), our result has to be the **green line**.
- However, our mathematical result is **red line**!!



Application:
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- Based on "Copenhagen Interpretation"
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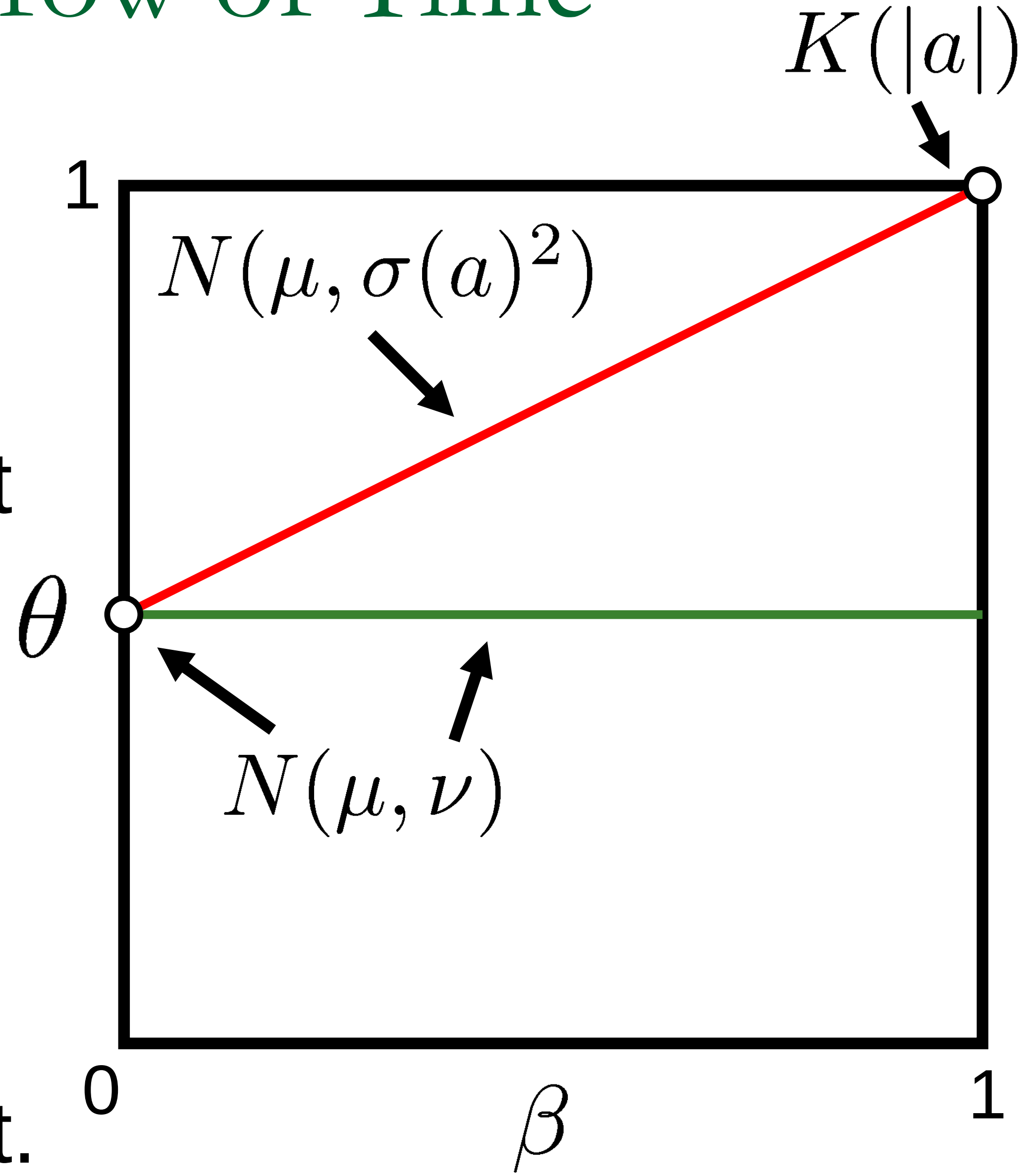
~~Conjecture (H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems*)~~

~~**The dynamics of the projective measurement can be taken as the Markov process.**~~

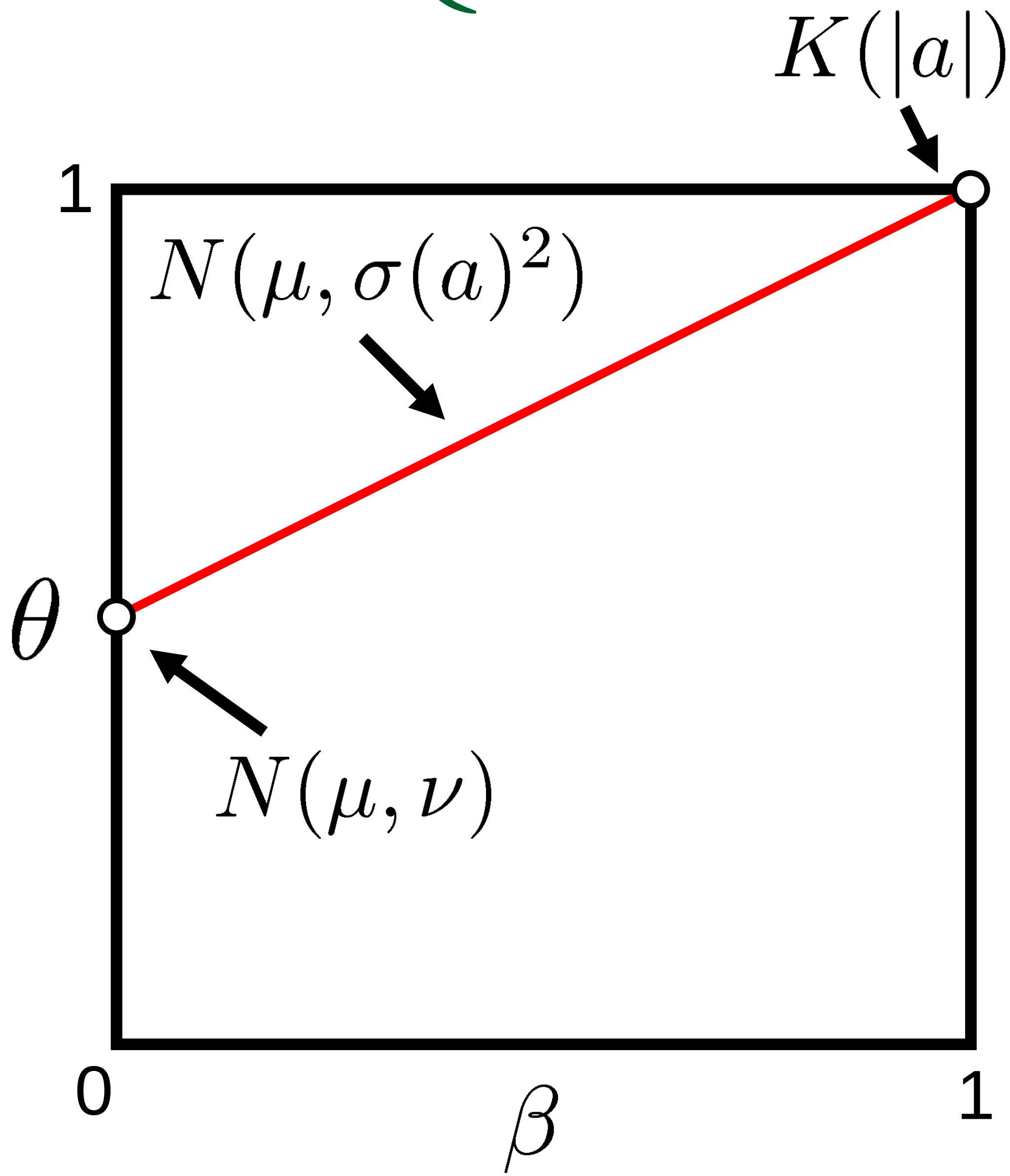
In other words, Arrow of Time

- According to the "Copenhagen interpretation", the projective measurement produces the arrow of time.
- Our result supports to gradually emerge the arrow of time by the projective measurement.



Summary of Decoherence QW

- There exist the quantification of randomness.
- The dynamics of the projective measurement cannot be taken as the Markov process.



Conclusions

- Introduction to Quantum Walk
  - Definition and Important Properties
    - In the case of the quantum walk with the multi-level coin, there appears in the **localization**.
- Decoherence Quantum Walk
  - Weak Limit Theorem
  - Quantification of Randomness
  - Application to Quantum Measurement Problem

Open Questions

- Physics
  - Many-Body System and Quantum Walk?
- Mathematics
  - Quantum Probability and Quantum Walk?
  - Quantum Ito Formula??
- Information
  - Other Useful Applications?
- Other
  - Meaning of Quantum Analogue?

Thank you for your attention.