Entanglement
in identical particle systems

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量子科学における双対性とスケール @基研

Problem

$|\psi_1\rangle \otimes |\psi_2\rangle$

- Normally, a state is called entangled iff it can't be written as a product state.
- Any fermionic state can't be written as a product state.

→ All fermionic state is entangled !

Preceding study

- Schliemann *et. al.* (2001)
  - Slater number, Schmidt number (←Schmidt decomposition)
- Ghirardi *et. al.* (2002)
  - Complete set of property (← EPR reality)

- Our work: We extend Ghirardi's framework to be able to treat arbitrary subsystems in fermionic sytems.
  - We introduce orthogonal subspaces.

→quant-ph/0910.1658

Outline

- Normally, a state is called entangled iff it can't be written as a $\boxed{\text{product}}$ state.
- Any fermionic state can't be written as a product state.

→ All fermionic state is entangled !

No

No

Decomposition of expectation values

Bell inequality (CHSH inequality)

Need for a structure of orthogonal subspaces

Reasonable extention to fermionic systems

Decomposition of expectation values in distinguishable particle systems

- $O_1 \otimes O_2 = (O_1 \otimes \mathbb{1})(\mathbb{1} \otimes O_2)$

Operator acting
on one particle

Decomposition of expectation values in distinguishable particle systems

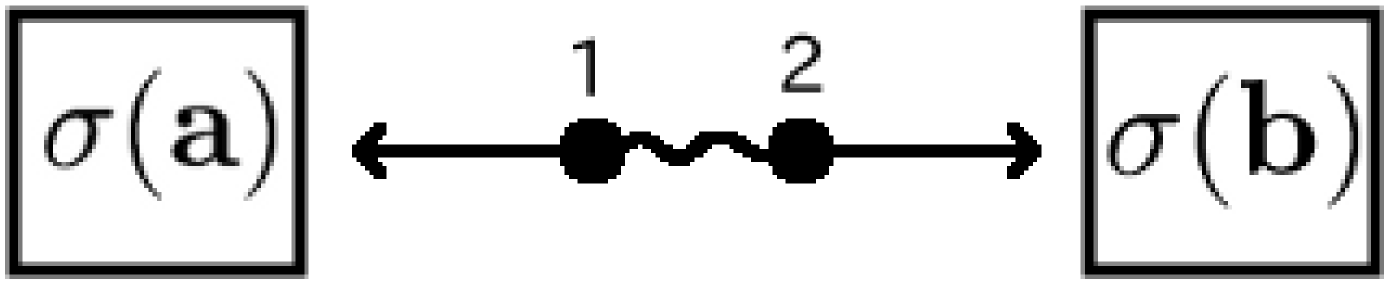
- $O_1 \otimes O_2 = (O_1 \otimes \mathbb{1})(\mathbb{1} \otimes O_2)$
- $|\psi\rangle$ can be written as a product state.

$\Longleftrightarrow$ Decomposition of expectation values

$$\langle\psi| O_1 \otimes O_2 |\psi\rangle$$
$$= \langle\psi| O_1 \otimes \mathbb{1} |\psi\rangle \langle\psi| \mathbb{1} \otimes O_2 |\psi\rangle$$

$\Longrightarrow$ Same result as in local realism

Setting
in distinguishable particle system



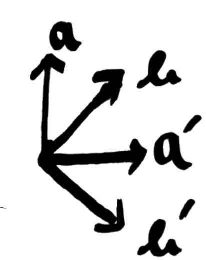
- CHSH $\qquad (\sigma(\mathbf{a}) := \mathbf{a}\cdot\sigma)$

$$O_{CHSH} = \sigma(\mathbf{a})\otimes\sigma(\mathbf{b}) + \sigma(\mathbf{a}')\otimes\sigma(\mathbf{b})$$
$$+\sigma(\mathbf{a})\otimes\sigma(\mathbf{b}') - \sigma(\mathbf{a}')\otimes\sigma(\mathbf{b}')$$

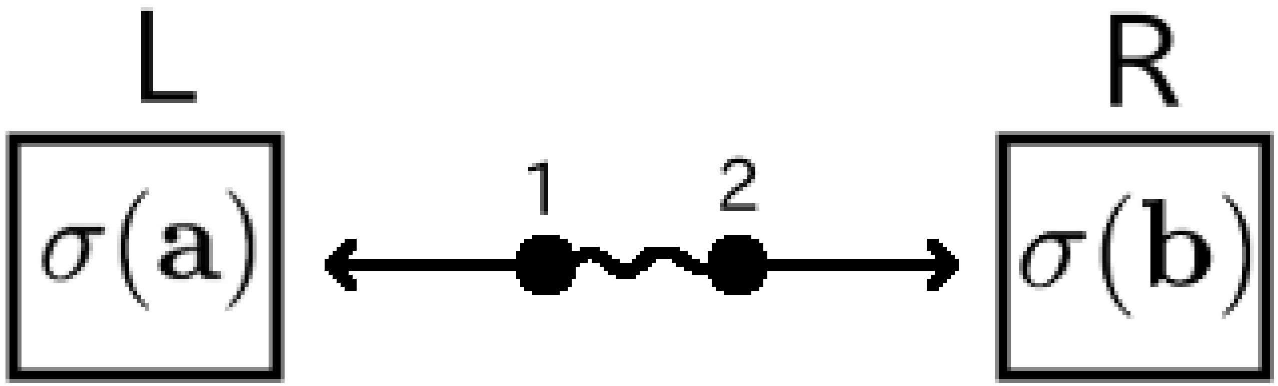
- Local realism $|\sigma(\mathbf{a})| \le 1 \Rightarrow |O_{CHSH}| \le 2$

- Quantum mechanics $\quad \langle\psi| O_{CHSH} |\psi\rangle = 2\sqrt{2}$

singlet



Setting (detail) in distinguishable particle system



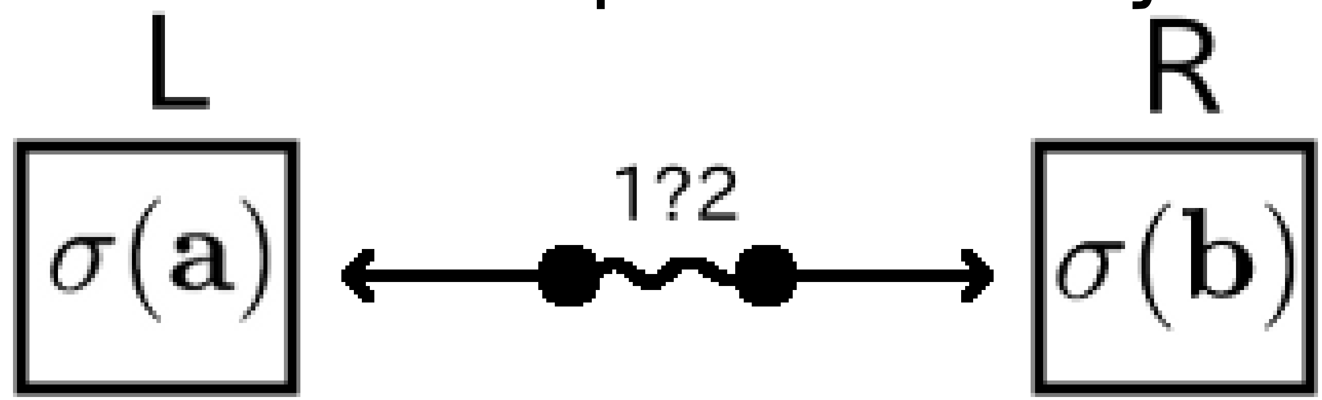
- Operator

$$\sigma(\mathbf{a}) \otimes \sigma(\mathbf{b}) \Rightarrow \sigma(\mathbf{a}) \left|L\right\rangle \left\langle L\right| \otimes \sigma(\mathbf{b}) \left|R\right\rangle \left\langle R\right|$$

- State

$$\left|\psi\right\rangle \Rightarrow \frac{1}{\sqrt{2}} \left(\left|\uparrow, L\right\rangle_1 \left|\downarrow, R\right\rangle_2 - \left|\downarrow, L\right\rangle_1 \left|\uparrow, R\right\rangle_2 \right)$$

Setting
in fermionic particle system

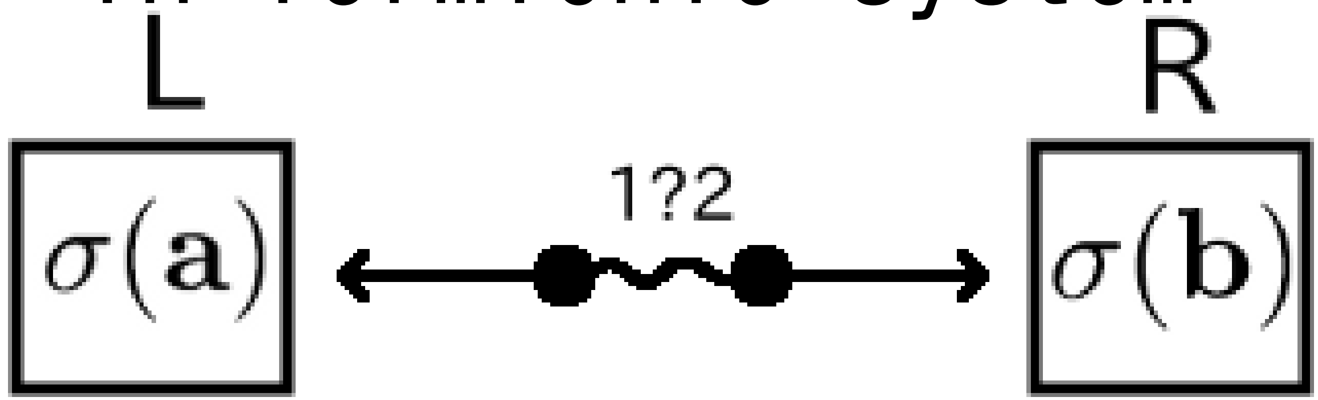


- Operator

$$\sigma(\mathbf{a})\,|L\rangle\langle L| \otimes \sigma(\mathbf{b})\,|R\rangle\langle R|$$

$$\Rightarrow \left\{ \begin{array}{l} \underbrace{\sigma(\mathbf{a})\,|L\rangle\langle L|}_{1} \otimes \underbrace{\sigma(\mathbf{b})\,|R\rangle\langle R|}_{2} + (1 \leftrightarrow 2) \\ \boxed{A\sigma(\mathbf{a})\,|L\rangle\langle L| \otimes \sigma(\mathbf{b})\,|R\rangle\langle R|\,A} \end{array} \right.$$

Setting in fermionic system



- Operator

O_L

$$\sigma(\mathbf{a})\,|L\rangle\langle L| \otimes \sigma(\mathbf{b})\,|R\rangle\langle R|$$

O_R

$$\Rightarrow\ A\sigma(\mathbf{a})\,|L\rangle\langle L| \otimes \sigma(\mathbf{b})\,|R\rangle\langle R|\,A$$

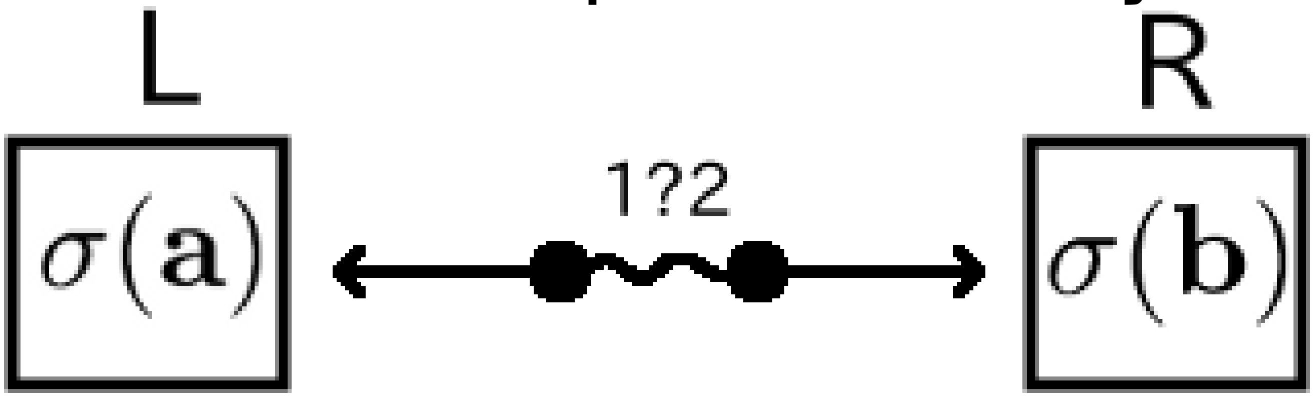
- Operator acting on one particle

$\mathbb{1}_R$

$$\Rightarrow \begin{cases} A\sigma(\mathbf{a})\,|L\rangle\langle L| \otimes \mathbb{1}\,|R\rangle\langle R|\,A \\ A\mathbb{1}\,|L\rangle\langle L| \otimes \sigma(\mathbf{b})\,|R\rangle\langle R|\,A \end{cases}$$

$\mathbb{1}_L$

Setting in fermionic particle system



- Operator

  $O_L \otimes O_R$

  $$\Rightarrow AO_L \otimes O_R A$$

- Operator acting on one particle

  $$\Rightarrow \begin{cases} AO_L \otimes \mathbb{1}_R A \\ A\mathbb{1}_L \otimes O_R A \end{cases}$$

$$\Rightarrow \quad \begin{aligned} & AO_L \otimes O_R A \\ & = AO_L \otimes \mathbb{1}_R A \\ & \quad \times A\mathbb{1}_L \otimes O_R A \end{aligned}$$

Relation

Distinguishable

Operator

$$O_L \otimes O_R \; (=: M)$$

$$O_L \otimes \mathbb{1}_R \; (=: M_1)$$

$$\mathbb{1}_L \otimes O_R \; (=: M_2)$$

$$\Rightarrow M = M_1 M_2$$

Decomposition

$$\langle\psi| M |\psi\rangle$$

$$= \langle\psi| M_1 |\psi\rangle \langle\psi| M_2 |\psi\rangle$$

Fermionic

Operator

$$AO_L \otimes O_R A \; (=: M)$$

$$AO_L \otimes \mathbb{1}_R A \; (=: M_1)$$

$$A\mathbb{1}_L \otimes O_R A \; (=: M_2)$$

$$\Rightarrow M = M_1 M_2$$

Decomposition

$$\langle\psi| M |\psi\rangle$$

$$= \langle\psi| M_1 |\psi\rangle \langle\psi| M_2 |\psi\rangle$$

Examples in fermionic system

- State
$$\begin{cases} |\psi_1\rangle = A\,|\alpha, L\rangle\,|\beta, R\rangle \\ |\psi_2\rangle = A\left(|\alpha, L\rangle\,|\beta, R\rangle - |\beta, L\rangle\,|\alpha, R\rangle\right) \end{cases}$$

- Decom-position
$$\begin{cases} \langle\psi_1|\,M\,|\psi_1\rangle = \langle\psi_1|\,M_1\,|\psi_1\rangle\,\langle\psi_1|\,M_2\,|\psi_1\rangle \\ \langle\psi_2|\,M\,|\psi_2\rangle \neq \langle\psi_2|\,M_1\,|\psi_2\rangle\,\langle\psi_2|\,M_2\,|\psi_2\rangle \end{cases}$$

- CHSH $(\alpha = \uparrow, \beta = \downarrow)$

$$\langle\psi_2|\,M_{CHSH}\,|\psi_2\rangle = 2\sqrt{2}$$

Relation

Distinguishable

Operator

$$O_L \otimes O_R \; (=: M)$$

$$O_L \otimes \mathbb{1}_R \; (=: M_1)$$

$$\mathbb{1}_L \otimes O_R \; (=: M_2)$$

Decomposition

$$\langle\psi| M |\psi\rangle$$
$$= \langle\psi| M_1 |\psi\rangle \langle\psi| M_2 |\psi\rangle$$

Separable state

$$|\psi\rangle = |\alpha, L\rangle |\beta, R\rangle$$

Fermionic

Operator

$$AO_L \otimes O_R A \; (=: M)$$

$$AO_L \otimes \mathbb{1}_R A \; (=: M_1)$$

$$A\mathbb{1}_L \otimes O_R A \; (=: M_2)$$

Decomposition

$$\langle\psi| M |\psi\rangle$$
$$= \langle\psi| M_1 |\psi\rangle \langle\psi| M_2 |\psi\rangle$$

Separable state

$$|\psi\rangle = A |\alpha, L\rangle |\beta, R\rangle$$

Key points

- $A\left|\alpha, L\right\rangle_1 \left|\beta, R\right\rangle_2$ is not a product state. However, expectation values for this state are decomposed.
- L,R play an essential role in the decomposition.
- When expectation values of a state are decomposed, it can’t violate CHSH inequarity.

Reconsideration of fermionic particle system (2 particle)

- Orthogonal subspaces $(V_1 \perp V_2)$

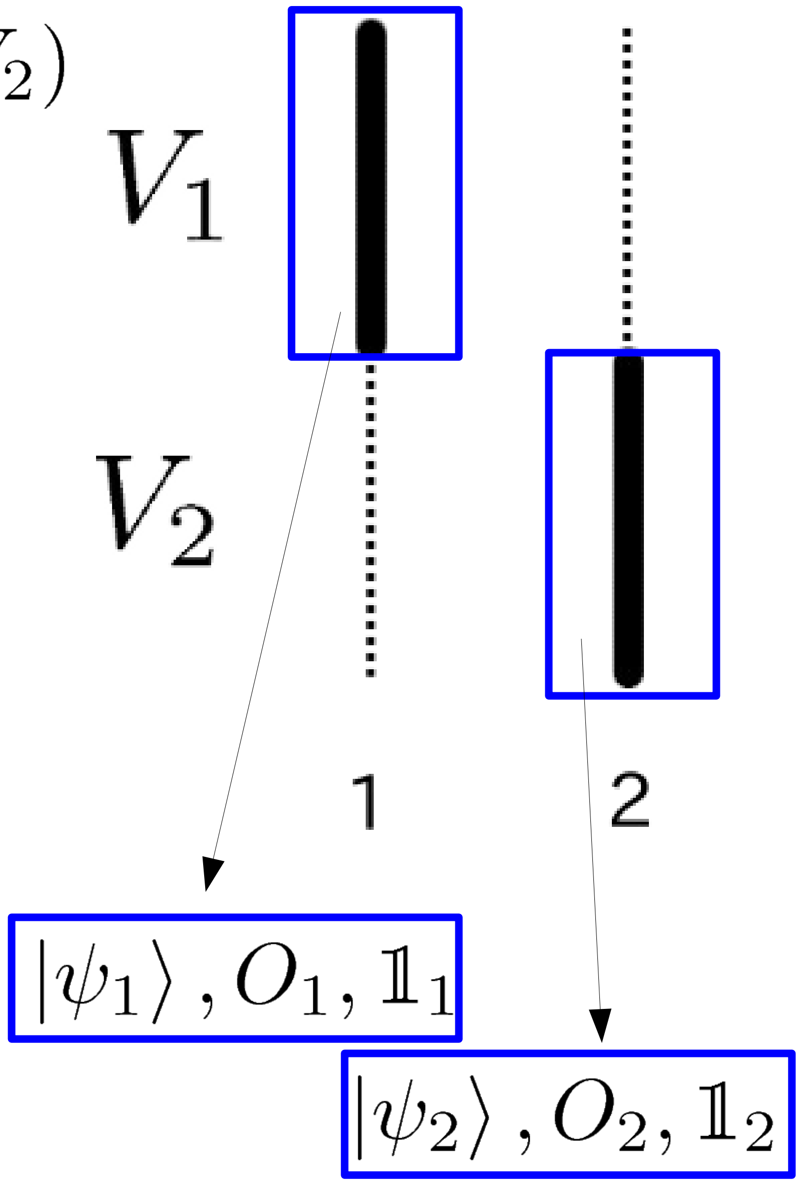
$$V_1 = \mathrm{span}\{|\uparrow, L\rangle, |\downarrow, L\rangle\}$$
$$V_2 = \mathrm{span}\{|\uparrow, R\rangle, |\downarrow, R\rangle\}$$

- Operation

$$AO_1 \otimes O_2 A$$
$$AO_1 \otimes \mathbb{1}_2 A$$
$$A\mathbb{1}_1 \otimes O_2 A$$

- Separable state

$$|\psi\rangle = A|\psi_1\rangle|\psi_2\rangle$$



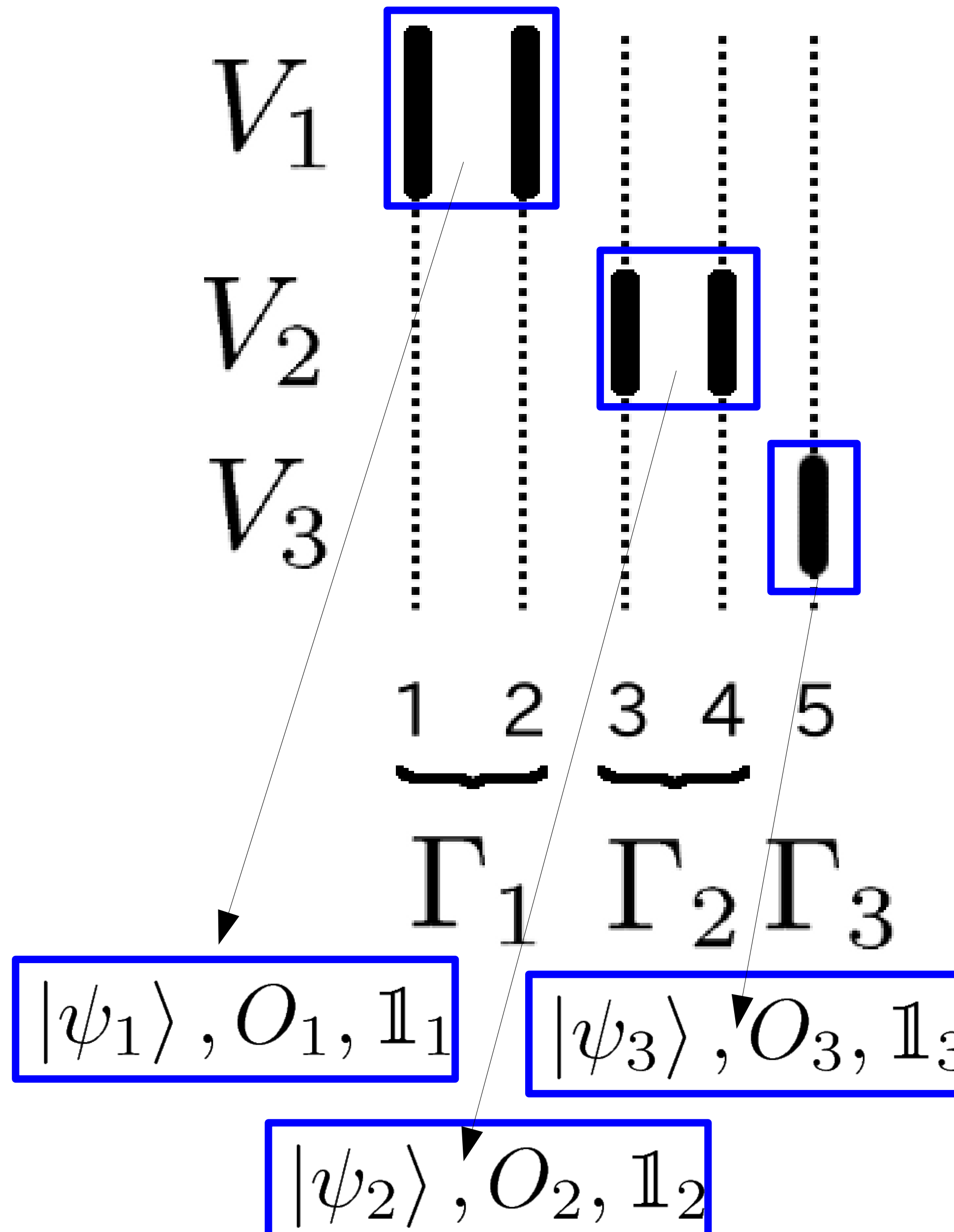
Generalization

- Orthogonal subspaces $V_1 \perp V_2 \perp V_3$
- Operation

$$AO_1 \otimes O_2 \otimes O_3 A \; (=: M)$$
$$AO_1 \otimes \mathbb{1}_2 \otimes \mathbb{1}_3 A \; (=: M_1)$$
$$A\mathbb{1}_1 \otimes O_2 \otimes \mathbb{1}_3 A \; (=: M_2)$$
$$A\mathbb{1}_1 \otimes \mathbb{1}_2 \otimes O_3 A \; (=: M_3)$$
$$M = M_1 M_2 M_3$$

- Separable state

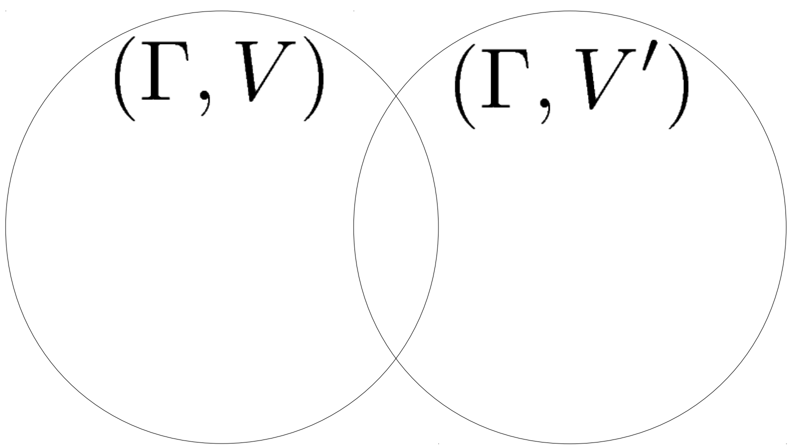
$$\langle\psi| M |\psi\rangle$$
$$= \langle\psi| M_1 |\psi\rangle \langle\psi| M_2 |\psi\rangle \langle\psi| M_3 |\psi\rangle$$
$$|\psi\rangle = A |\psi_1\rangle |\psi_2\rangle |\psi_3\rangle$$



Discussion

- A structure of orthogonal subspaces $\Longleftarrow$ A setting of experiment

- Arbitrariness of orthogonal subspaces V



Summary

- In indentical particle systems, it is not appropriate to define entanglement using direct product.
- Introducing a structure of orthogonal subspaces, we can treat identical particle systems in the same way as distinguishable particle systems.
- There is arbitrariness to select a structure of orthogonal subspaces.
- A structure of orthogonal subspaces can be interpretated as a setting of experiment.