Exact RG flow and UV/IR duality in quantum graphs

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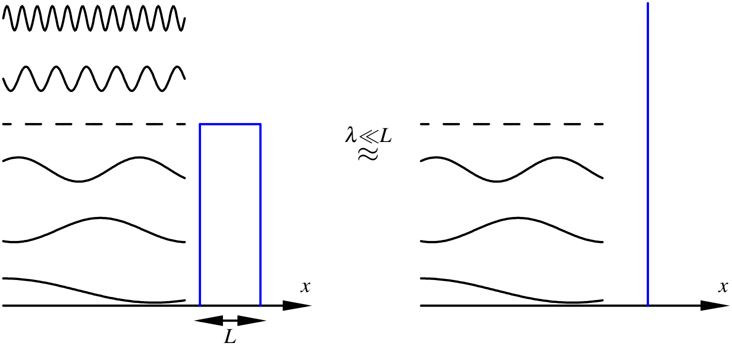
Introduction ①

- Slowly moving particle cannot resolve the structure of a short-ranged scatterer (e.g. an impurity or a defect).
- (Much) below a physical cutoff scale L, any short-ranged interaction can be approximated by a point interaction.



Introduction ②: trivial example

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x), \quad V(x) = \begin{cases} \frac{2g}{L}, & -\frac{L}{2} < x < \frac{L}{2} \\ 0, & \text{otherwise} \end{cases}$$

- (Much) below the cutoff scale L, reflection and transmission coefficients for a particle of momentum k are given by

$$R(k;g) = \frac{g}{ik - g}, \quad T(k;g) = \frac{ik}{ik - g}.$$

- On length scale $\gg L$, we require that the physics (i.e. function forms of R and T) should not change as we change the momentum scale $k \mapsto k\mathrm{e}^t$. This requirement is expressed as

$$R(k\mathrm{e}^t; g) = R(k; \bar{g}(t)) \quad \text{and} \quad T(k\mathrm{e}^t; g) = T(k; \bar{g}(t)),$$

from which we find the running coupling constant

$$\bar{g}(t) = g\mathrm{e}^{-t}.$$

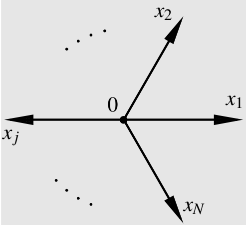
Today's talk

- spinless one-particle quantum mechanics on star graph
- absence of long-ranged interaction (such as Coulomb force)
- long-wavelength limit

$\Rightarrow$ theory space = parameter space of $U(N)$

Quantum particle on a star graph

One particle quantum mechanics on star graph with N bonds



- jth bond = jth half-line ($0 < x_j < \infty$)
- vertex = junction point $x_j = 0$, $^\forall j$
- wave function

$$\psi(x) = \begin{cases} \psi_1(x_1), & x \in \text{1st bond}, \\ \quad\vdots & \\ \psi_N(x_N), & x \in N\text{th bond}. \end{cases}$$

Schrödinger equation (in the units $\hbar = 2m = 1$)

$$H\psi(x) = E\psi(x), \quad H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}, \quad x \neq 0$$

Kirchhoff's law for boundary conditions ①

Requirement: probability conservation at the vertex

$$\sum_{j=1}^{N} j_j(0) = 0, \quad j_j(x_j) := -i\left[\psi_j'^*(x_j)\psi_j(x_j) - \psi_j^*(x_j)\psi_j'(x_j)\right]$$

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$$\Leftrightarrow \begin{pmatrix} \psi_1(0) \\ \vdots \\ \psi_N(0) \end{pmatrix}^{\dagger} \cdot \begin{pmatrix} \psi_1'(0) \\ \vdots \\ \psi_N'(0) \end{pmatrix} = \begin{pmatrix} \psi_1'(0) \\ \vdots \\ \psi_N'(0) \end{pmatrix}^{\dagger} \cdot \begin{pmatrix} \psi_1(0) \\ \vdots \\ \psi_N(0) \end{pmatrix}$$

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$$\Leftrightarrow \left|\begin{pmatrix}\psi_1(0)\\ \vdots \\ \psi_N(0)\end{pmatrix} - iL_0 \begin{pmatrix}\psi_1'(0)\\ \vdots \\ \psi_N'(0)\end{pmatrix}\right|^2 = \left|\begin{pmatrix}\psi_1(0)\\ \vdots \\ \psi_N(0)\end{pmatrix} + iL_0 \begin{pmatrix}\psi_1'(0)\\ \vdots \\ \psi_N'(0)\end{pmatrix}\right|^2$$

(L_0: arbitrary length scale (a.k.a scale anomaly))

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$$\Leftrightarrow \left| \begin{pmatrix} \psi_1(0) \\ \vdots \\ \psi_N(0) \end{pmatrix} - iL_0 \begin{pmatrix} \psi_1'(0) \\ \vdots \\ \psi_N'(0) \end{pmatrix} \right|^2 = \left| \begin{pmatrix} \psi_1(0) \\ \vdots \\ \psi_N(0) \end{pmatrix} + iL_0 \begin{pmatrix} \psi_1'(0) \\ \vdots \\ \psi_N'(0) \end{pmatrix} \right|^2$$

(L_0: arbitrary length scale (a.k.a scale anomaly))

$$\Leftrightarrow \begin{pmatrix} \psi_1(0) \\ \vdots \\ \psi_N(0) \end{pmatrix} - iL_0 \begin{pmatrix} \psi_1'(0) \\ \vdots \\ \psi_N'(0) \end{pmatrix} = U \left[\begin{pmatrix} \psi_1(0) \\ \vdots \\ \psi_N(0) \end{pmatrix} + iL_0 \begin{pmatrix} \psi_1'(0) \\ \vdots \\ \psi_N'(0) \end{pmatrix} \right]$$

$(U \in U(N))$

Kirchhoff's law for boundary conditions ②

- $U(N)$ family of boundary conditions

$$(\mathbb{1}-U)\begin{pmatrix}\psi_1(0)\\ \vdots \\ \psi_N(0)\end{pmatrix} - iL_0(\mathbb{1}+U)\begin{pmatrix}\psi_1'(0)\\ \vdots \\ \psi_N'(0)\end{pmatrix} = \vec{0}, \quad U \in U(N).$$

- spectral decomposition of U

$$U = \sum_{j=1}^{N} \mathrm{e}^{i\alpha_j} P_j, \quad P_j : \text{projection operator}$$

- $U(N)$ family of boundary conditions boils down to the N independent conditions

$$P_j \left[\begin{pmatrix}\psi_1(0)\\ \vdots \\ \psi_N(0)\end{pmatrix} + L_j \begin{pmatrix}\psi_1'(0)\\ \vdots \\ \psi_N'(0)\end{pmatrix} \right] = \vec{0}, \quad L_j := L_0 \cot\frac{\alpha_j}{2}$$

Example: $N=2$ (junction of two half-lines = line)

$$\frac{\mathbb{1}+\vec{e}_j\cdot\vec{\sigma}}{2}\left[\begin{pmatrix}\psi_1(0)\\ \psi_2(0)\end{pmatrix}+L_j\begin{pmatrix}\psi_1'(0)\\ \psi_2'(0)\end{pmatrix}\right]=\vec{0},\quad \vec{e}_2=-\vec{e}_1 : \text{unit vector}$$

Case $\vec{e}_1=(0,0,1)$: separated subfamily

$$\psi_1(0)+L_1\psi_1'(0)=0,$$
$$\psi_2(0)+L_2\psi_2'(0)=0.$$

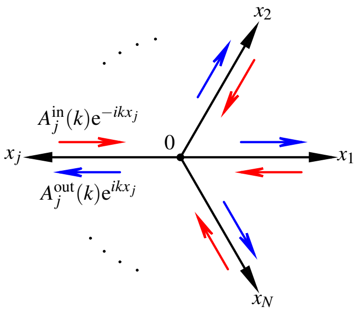
- $L_{1,2}=0$ ($\alpha_{1,2}=\pi$): Dirichlet BC
- $L_{1,2}=\infty$ ($\alpha_{1,2}=0$): Neumann BC

Case $\vec{e}_1=(1,0,0)$: parity invariant subfamily

$$\psi_1(0)+\psi_2(0)+L_1(\psi_1'(0)+\psi_2'(0))=0,$$
$$\psi_1(0)-\psi_2(0)+L_2(\psi_1'(0)-\psi_2'(0))=0.$$

- $L_2=0$ ($\alpha_2=\pi$): δ-function potential $V(x)=\frac{1}{L_1}\delta(x)$

S-matrix on star graph



- positive energy solution

$$\psi_j(x_j;k) = A_j^{\rm in}(k)\mathrm{e}^{-ikx_j} + A_j^{\rm out}(k)\mathrm{e}^{ikx_j}$$

- evolution map between "in-state" and "out-state":

$$\begin{pmatrix} A_1^{\rm out} \\ \vdots \\ A_N^{\rm out} \end{pmatrix} = S(kL_0) \begin{pmatrix} A_1^{\rm in} \\ \vdots \\ A_N^{\rm in} \end{pmatrix},$$

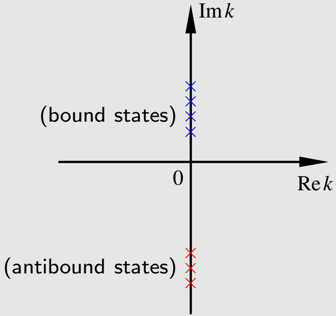
where the S-matrix $S(kL_0)$ is given by

$$S(kL_0) = \begin{pmatrix} R_1(kL_0) & T_{12}(kL_0) & \cdots & T_{1N}(kL_0) \\ T_{21}(kL_0) & R_2(kL_0) & \cdots & T_{2N}(kL_0) \\ \vdots & \vdots & \ddots & \vdots \\ T_{N1}(kL_0) & T_{N2}(kL_0) & \cdots & R_N(kL_0) \end{pmatrix} = \sum_{j=1}^{N} \frac{ikL_j - 1}{ikL_j + 1} P_j.$$

Bound states and antibound (or virtual) states

- bound (antibound) state = simple pole of $S(kL_0)$ lying on the **positive** (**negative**) imaginary k-axis
- bound/antibound state energy at a pole $k = i/L_l$:

$$E_{B_l} = \left(\frac{i}{L_l}\right)^2 = -\frac{1}{L_l^2}$$

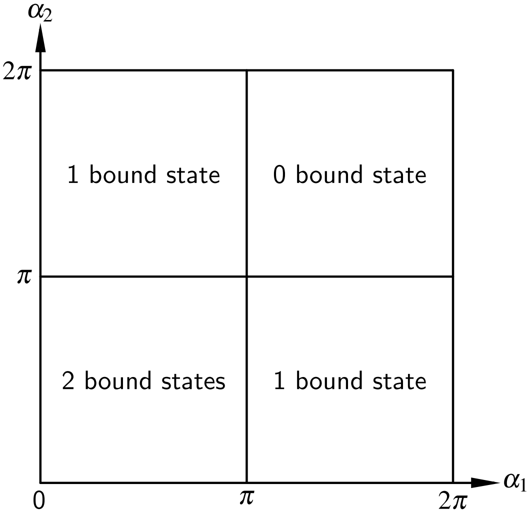


- bound/antibound state wave function at a pole $k = i/L_l$:

$$\psi_{B_l,j}(x_j) \propto \exp\left(-\frac{x_j}{L_l}\right) \quad \left(L_l = L_0 \cot\frac{\alpha_l}{2}, \quad L_0 > 0\right)$$

$$\Rightarrow \begin{cases} \text{normalizable bound state} & \text{for } 0 < \alpha_l < \pi \\ \text{non-normalizable antibound state} & \text{for } \pi < \alpha_l < 2\pi \end{cases}$$

Phase diagram for $N = 2$



Exact RG flow of point interaction ①

Renormalization group transformation

Since L_0 is an arbitrary parameter, any physical quantities must be independent of the choice of L_0. The lack of dependence of L_0 can be expressed as the invariance of the theory under the RG transformation

$$R_t : L_0 \mapsto \bar{L}(t) := L_0 \mathrm{e}^{-t}, \quad -\infty < t < \infty.$$

Renormalization group equation

Any change of L_0 must be equivalent to changes in the $U(N)$ parameters. This is expressed as the RG equation

$$S(kL_0; \alpha_j, P_j) = S(k\bar{L}(t); \bar{\alpha}_j(t), \bar{P}_j(t)),$$

or, equivalently,

$$S((k\mathrm{e}^t)L_0; \alpha_j, P_j) = S(kL_0; \bar{\alpha}_j(t), \bar{P}_j(t)).$$

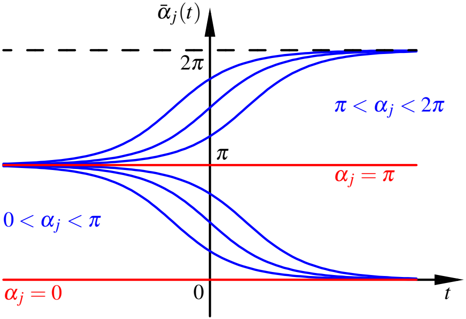
Exact RG flow of point interaction ②

Since k appears only in the combination $kL_j = kL_0 \cot\frac{\alpha_j}{2}$, the rescaling of k must be adjusted by the running of α_j:

$$kL_0 \cot\frac{\alpha_j}{2} \overset{k\to k\mathrm{e}^t}{\longmapsto} kL_0\left(\mathrm{e}^t \cot\frac{\alpha_j}{2}\right) = kL_0\left(\cot\frac{\bar{\alpha}_j(t)}{2}\right),$$

from which we find

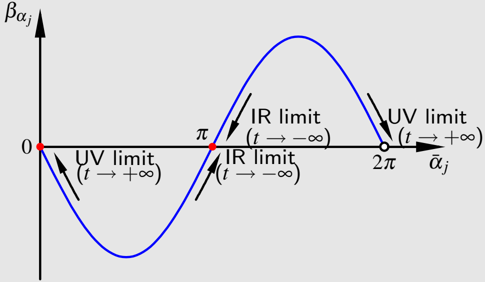
$$\bar{\alpha}_j(t) = 2\arctan\left(\mathrm{e}^{-t}\tan\frac{\alpha_j}{2}\right) \quad \text{and} \quad \bar{P}_j(t) = P_j.$$



Exact RG flow of point interaction ③

exact β-function

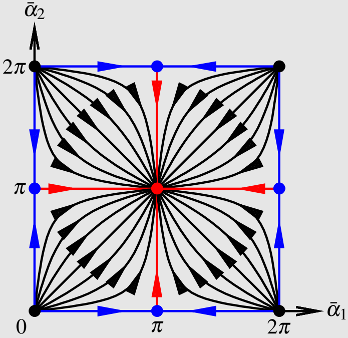
$$\beta_{\alpha_j}(\bar{\alpha}_j(t)) := \left.\frac{\partial \bar{\alpha}_j(t)}{\partial t}\right|_{\alpha_j, L_0} = -\sin \bar{\alpha}_j(t)$$



- $\alpha_j^* = 0$: UV fixed point; $\alpha_j^* = \pi$: IR fixed point
- 2^N fixed points on $T^N = \{(\alpha_1, \cdots, \alpha_N) \mid 0 \le \alpha_j < 2\pi\}$

Exact RG flow of point interaction ③

Example: $N = 2$



- UV (Neumann) fixed point •

$$U = S = \mathbb{1}$$
$$\psi_1'(0) = 0 = \psi_2'(0)$$

- IR (Dirichlet) fixed point •

$$U = S = -\mathbb{1}$$
$$\psi_1(0) = 0 = \psi_2(0)$$

- Fülöp-Tsutsui fixed point •

$$U = S = \begin{pmatrix} \cos\varphi & e^{-i\theta}\sin\varphi \\ e^{i\theta}\cos\varphi & -\cos\varphi \end{pmatrix}$$
$$\psi_2(0) = e^{i\theta}\tan\frac{\varphi}{2}\psi_1(0), \quad \psi_2'(0) = -e^{i\theta}\cot\frac{\varphi}{2}\psi_1'(0)$$

Duality

- elementary identity

$$\frac{1}{\cot\frac{\alpha_j}{2}} = \tan\frac{\alpha_j}{2} = -\cot\left(\frac{\alpha_j}{2} \pm \frac{\pi}{2}\right)$$

- identity of S-matrix eigenvalue

$$\frac{ikL_0\cot\frac{\alpha_j}{2} - 1}{ikL_0\cot\frac{\alpha_j}{2} + 1} = -\frac{i(kL_0)^{-1}\cot(\frac{\alpha_j}{2} \pm \frac{\pi}{2}) - 1}{i(kL_0)^{-1}\cot(\frac{\alpha_j}{2} \pm \frac{\pi}{2}) + 1}$$

- identity of S-matrix

$$S(kL_0;\alpha_j,P_j) = -S((kL_0)^{-1};\alpha_j \pm \pi, P_j)$$

which implies that

	$\{\alpha_j,P_j\}_{j=1}^N$		$\{\alpha_j \pm \pi, P_j\}_{j=1}^N$
$E>0$	UV regime IR regime	$\overset{\text{dual}}{\Leftrightarrow}$	IR regime UV regime
$E<0$	bound states antibound states	$\overset{\text{dual}}{\Leftrightarrow}$	antibound states bound states

Summary

- Exact RG flow of $U(N)$ family of point interactions
- Duality between the systems $\{\alpha_j, P_j\}$ and $\{\alpha_j \pm \pi, P_j\}$
  - scattering states: high energy $\Leftrightarrow$ low energy
  - bound state $\Leftrightarrow$ antibound state