Mini-workshop in Tokyo Tech.
11 July 2008

# Kibble-Zurek argument on quantum annealing and simulated annealing

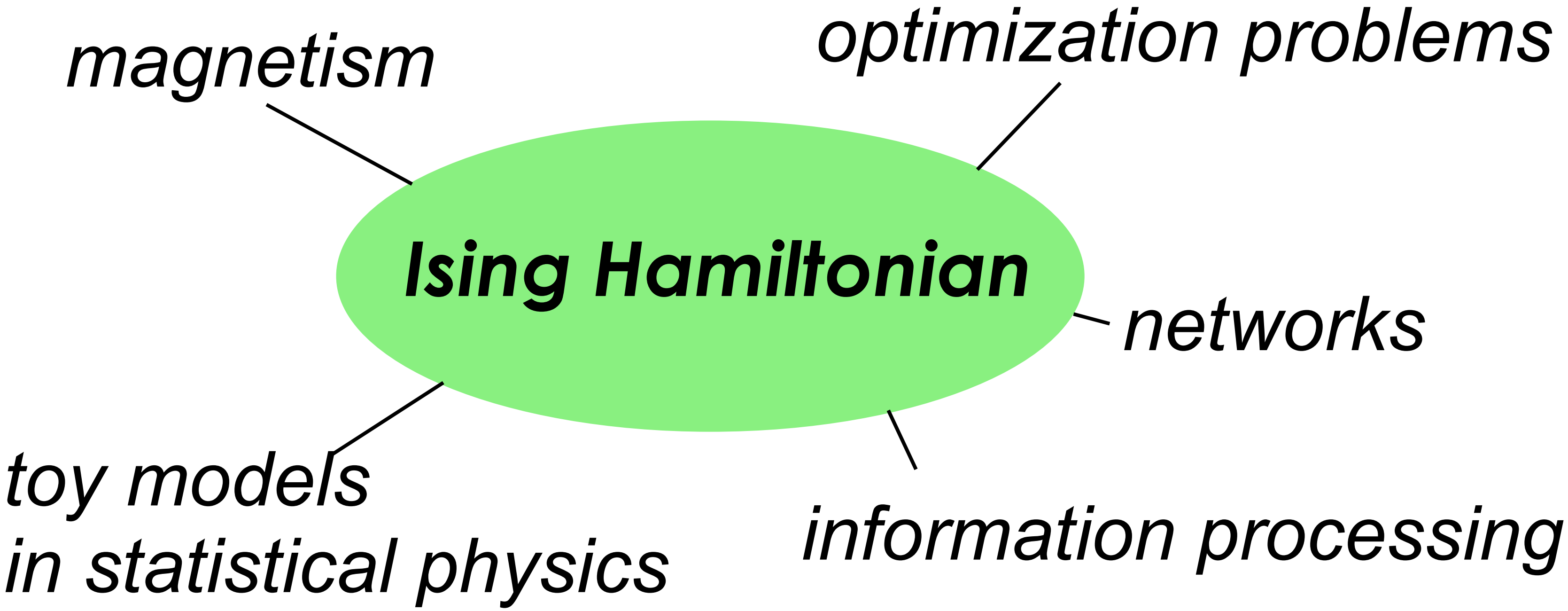
Sei Suzuki

*Aoyama Gakuin University*

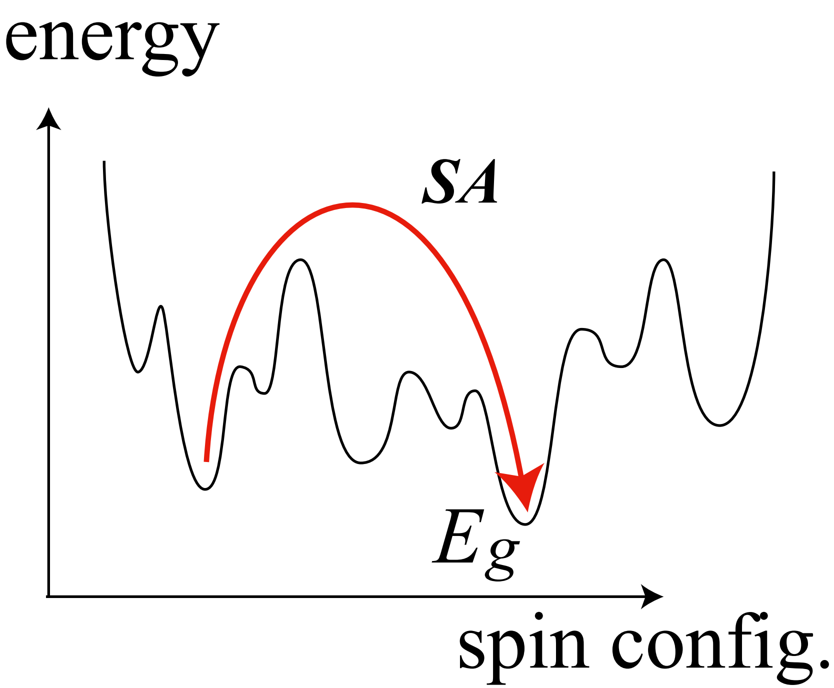
## 1. Introduction

Ground state problem of Ising Hamiltonian

$$\mathcal{H} = -\sum_i J_i \sigma_i - \sum_{ij} J_{ij} \sigma_i \sigma_j - \cdots$$



## Simulated (thermal) annealing (SA)



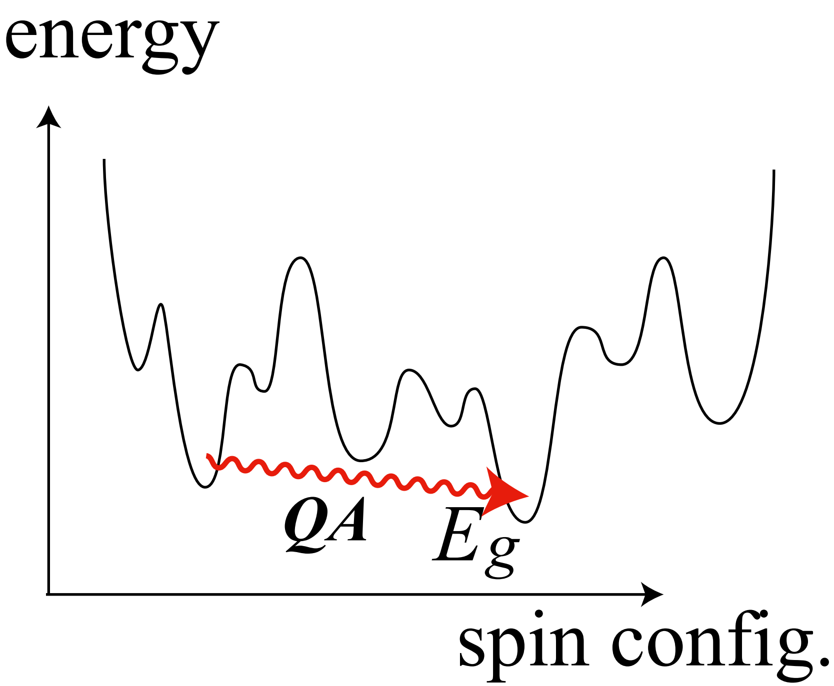

- thermal fluctuation
- equilibrium at high temp.

$\downarrow$ slow cooling

ground state

**Quantum annealing (QA)** (Finnila *et al.* '94, Kadowaki-Nishimori '98)

Quantum adiabatic computation (Farhi *et al.* '01)




- quantum fluctuation

  ⇒tunneling effect

- an available ground state

  ↓ adiabatic evolution

  ground state in question

## More about QA

$\mathcal{H}_{\text{pot}}$ : Ising Hamiltonian

$\mathcal{H}_{\text{kin}}$ : kinetic Hamiltonian (transverse field, typically)

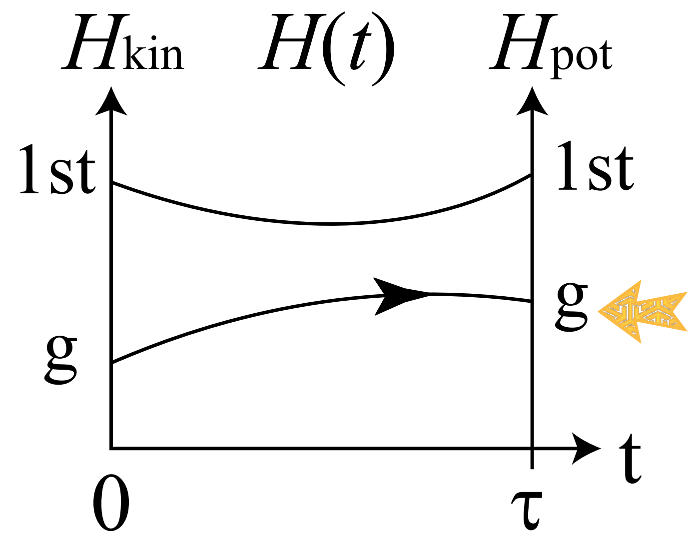
$$\left[\mathcal{H}_{\text{kin}}, \mathcal{H}_{\text{pot}}\right] \neq 0$$

time-dependent Hamiltonian :

$$\mathcal{H}(t) = f(t)\mathcal{H}_{\text{kin}} + g(t)\mathcal{H}_{\text{pot}}$$

$$\mathcal{H}(0) = \mathcal{H}_{\text{kin}}, \quad \mathcal{H}(\tau) = \mathcal{H}_{\text{pot}}$$

initial state: ground state of $\mathcal{H}(0)$

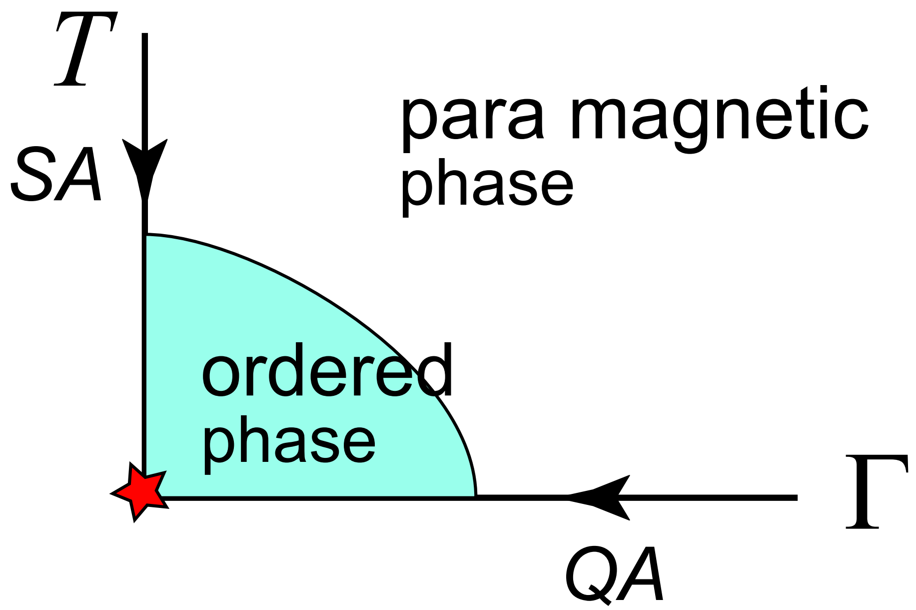



**quantum annealing** $=$ **quantum adiabatic computation**

## Phase diagram

temperature vs transverse field

$$\mathcal{H} = -\Gamma \sum_i \sigma_i^x + \mathcal{H}_{\text{Ising}}$$

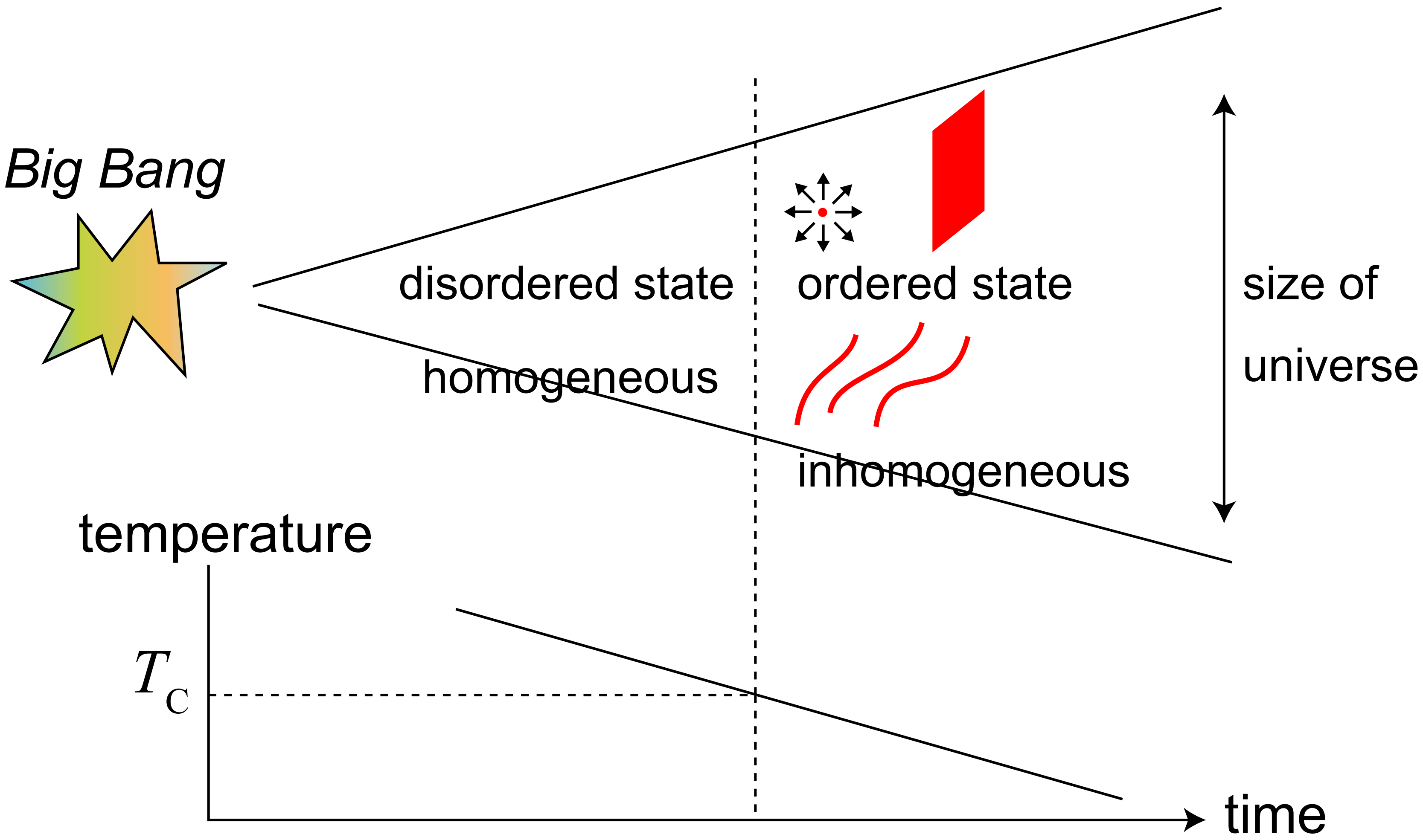



dynamics across phase transition

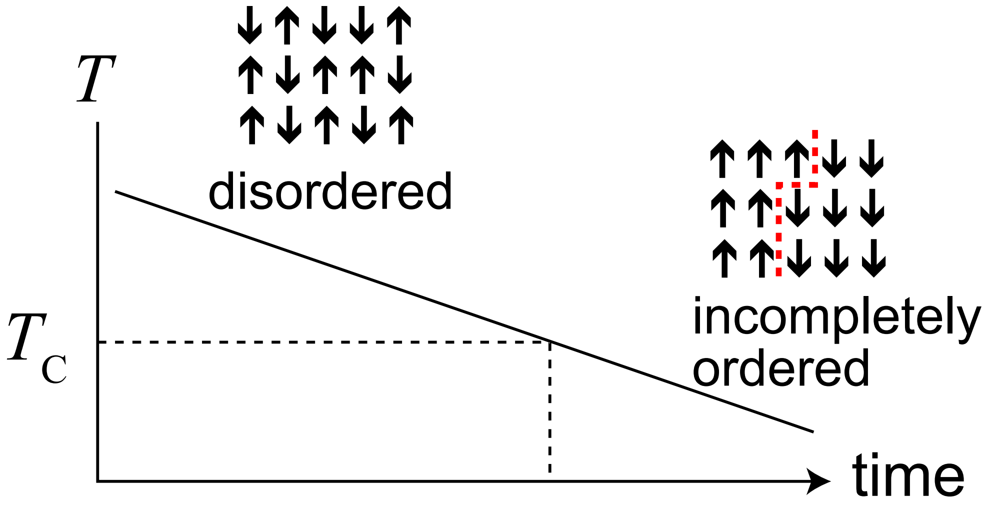
$\Leftarrow$ **Kibble-Zurek mechanism**

# Kibble-Zurek mechanism

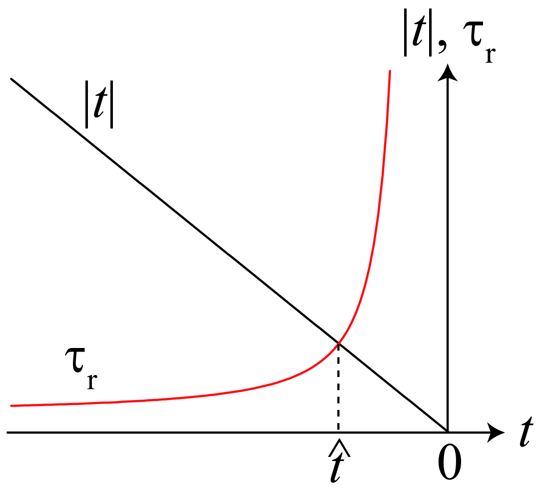
evolution of the universe



# Ising magnet




## Kibble-Zurek argument :

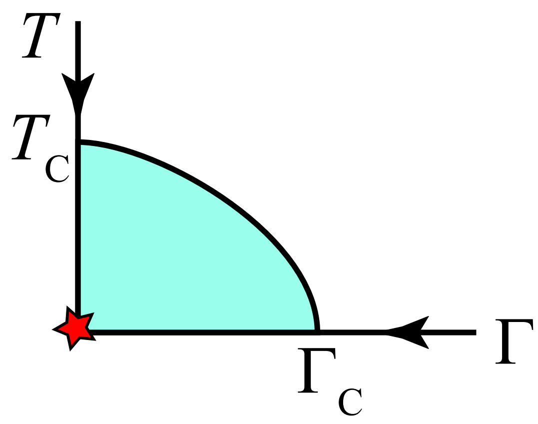


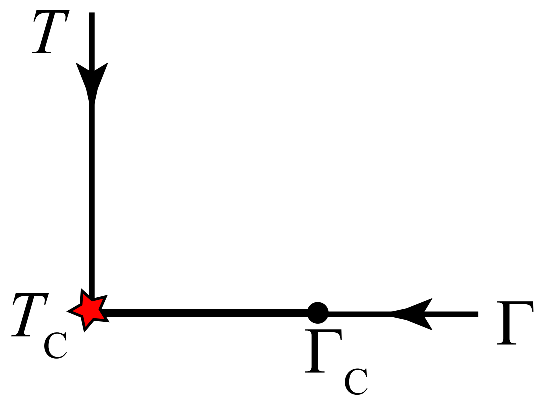

- ⋆ dimensionless temperature $\varepsilon \equiv (T - T_{\mathrm{C}})/T_{\mathrm{C}}$
- ⋆ correlation length: $\xi \propto \varepsilon^{-\nu}$; relaxation time: $\tau_r \propto \xi^z$
- ⋆ quenching schedule: $T = T_{\mathrm{C}}(1 - t/\tau) \Rightarrow \varepsilon = -t/\tau$
- ⋆ **a condition** ($\tau_{\mathrm{r}} =$ remaining time) : $\hat{\tau}_{\mathrm{r}} = |\hat{t}|$

$$\hat{\xi}^z \propto \tau \hat{\xi}^{-1/\nu} \Rightarrow \hat{\xi} \propto \tau^{\nu/(1+z\nu)}$$

Classical path and quantum path

Which path is better ?




Answer :

in 1D at least,

- both paths are equivalent (pure system)
- quantum path has an advantage (random system)

## 2. SA of pure Ising chain (SS)

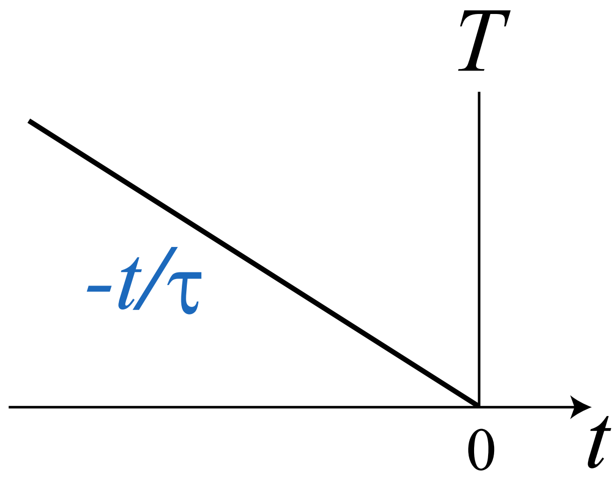
$$\mathcal{H} = -\sum_i \sigma_i \sigma_{i+1}$$

Assumption

- Glauber's dynamics
- cooling schedule :

$$T(t) = -t/\tau \qquad t : -\infty \to 0$$

- the initial state in equilibrium




How many kinks are there after cooling ?

$$|\uparrow\downarrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\uparrow\downarrow\uparrow\cdots\rangle \longrightarrow |\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow\downarrow\uparrow\uparrow\cdots\rangle$$

# Kibble-Zurek argument

criticality at $T = 0$

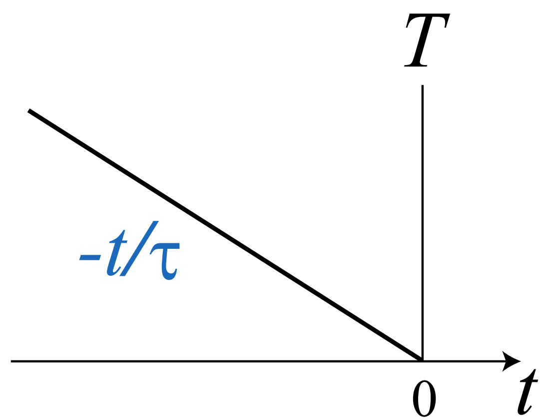
correlation length: $\xi \approx \frac{1}{2} e^{2/T}$

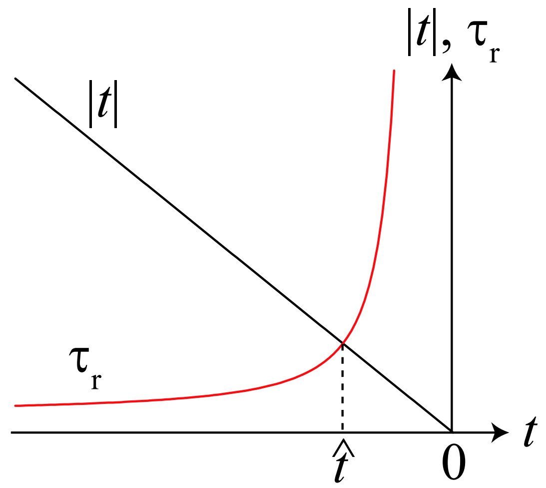
relaxation time: $\tau_r \approx \frac{1}{2} e^{4/T} \approx 2\xi^2$

**condition** ($\tau_r =$ remaining time):

$$\tau_r(T(\hat{t})) = |\hat{t}|$$

$$\Rightarrow \quad 2\hat{\xi}^2 = \frac{2\tau}{\ln(2\hat{\xi})} \quad \Rightarrow \quad \hat{\xi} = \sqrt{\frac{\tau}{\ln(2\hat{\xi})}} \approx \sqrt{\tau}$$




density of kinks after cooling

$$\rho \approx 1/\hat{\xi} \approx 1/\sqrt{\tau}$$

# 3. QA of pure Ising chain (Zurek *et al* '05)

$$\mathcal{H}(t) = -\Gamma(t)\sum_i \sigma_i^x - \sum_i \sigma_i^z \sigma_{i+1}^z$$

$$\Gamma(t) = -t/\tau \quad (t : -\infty \to 0)$$

- quantum phase transition at $\Gamma(t) = \Gamma_c = 1$ $(\varepsilon \equiv \Gamma(t) - 1)$
- correlation length: $\xi \approx \varepsilon^{-1}$
- excitation gap: $\Delta \approx \frac{1}{2}\varepsilon$
- "relaxation time": $\tau_r = 1/\Delta \approx 2\varepsilon^{-1} \approx 2\xi$
- **condition** ($\tau_r =$ remaining time)

$$\tau_r(\hat{\varepsilon}) = -\tau - \hat{t} \quad \Rightarrow \quad \hat{\varepsilon} = \sqrt{2/\tau} \quad \Rightarrow \quad \hat{\xi} \approx \sqrt{\tau/2}$$

density of kinks after quantum quenching

$$\rho \approx 1/\hat{\xi} \approx \sqrt{2/\tau}$$

## 4. SA of random Ising chain (SS)

$$\mathcal{H} = -\sum_i J_i \sigma_i \sigma_{i+1} \ , \quad J_i \in [0,1]$$

cooling schedule : $T = -t/\tau$ ; criticality at $T = 0$

correlation length: $\xi \approx 1/(T \ln 2)$

relaxation time (Dhar-Barma): $\tau_r \approx \frac{1}{2} e^{4/T} \approx \frac{1}{2} e^{(4 \ln 2)\xi}$

density of kinks and residual energy

$$\rho \approx 1/\hat{\xi} \sim 1/\ln\tau \quad , \quad \varepsilon_{res} \approx \frac{\pi^2}{24}\hat{T}^2 \sim 1/(\ln\tau)^2$$

# 5. QA of random Ising chain (Dziarmaga '06)

$$\mathcal{H}(t) = -\Gamma(t)\sum_i h_i \sigma_i^x - \sum_i J_i \sigma_i^z \sigma_{i+1}^z$$

$$\Gamma(t) = -t/\tau \quad (t : -\infty \to 0) \quad ; \quad h_i, J_i \in [0,1]$$

quantum phase transition at $\Gamma(t) = \Gamma_c = 1$ $(\varepsilon \equiv \Gamma(t) - 1)$

correlation length: $\xi \approx \varepsilon^{-2}$

"relaxation time": $\tau_r = 1/\Delta \approx \varepsilon^{-1/|\varepsilon|} \approx \xi^{\sqrt{\xi}/2}$

density of kinks after quantum quenching

$$\rho \approx 1/\hat{\xi} \approx 1/(\ln \tau)^2$$

Comparison of correlation length and relaxation time

Critical behaviors of $\tau_r$ are different.

| | | $\ln \xi$ | $\ln \tau_r$ |
|---|---|---|---|
| pure | c | $1/\varepsilon$ | $2 \ln \xi$ |
| | q | $-\ln \varepsilon$ | $\ln \xi$ |
| random | c | $-\ln \varepsilon$ | $\xi$ |
| | q | $-\ln \varepsilon$ | $\xi^{1/2}$ |

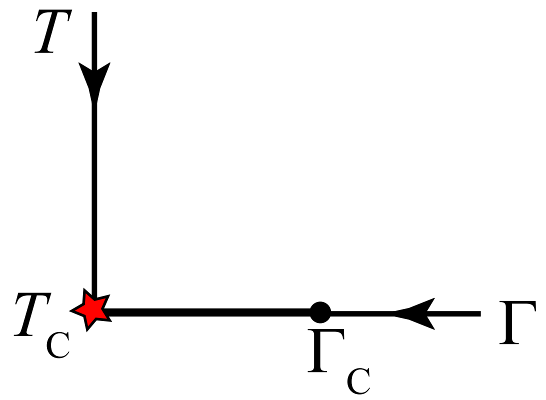
# Summary on 1D systems

| | | *QA* | *SA* |
|---|---|---|---|
| density of kinks | *pure* | $1/\tau^{1/2}$ (Dziarmaga) | $1/\tau^{1/2}$ |
| | *random* | $1/(\ln(\tau))^2$ (Dziarmaga, Tommaso *et al.*) | $1/\ln(\tau)$ |
| residual energy | *random* | $1/(\ln(\tau))^{3.4}$ (numerical study by Tommaso *et al.*) | $1/(\ln(\tau))^2$ (Huse-Fisher*) |

QA has an advantage over SA in the random case.

## Discussion (speculation for models in higher dimension)

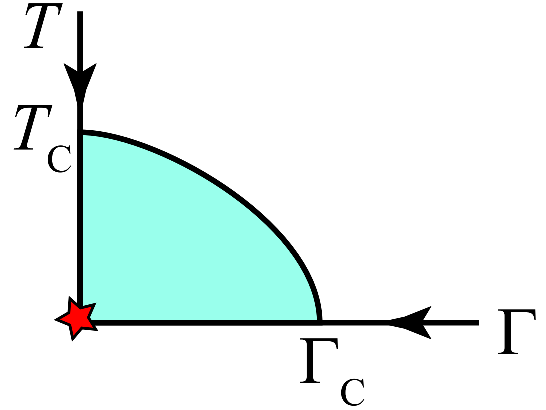
- **asymmetric** phase diagram in 1D models looks handicapped for QA




- models in higher dimension should be **symmetric**

  ⇒ fair for both QA and SA

  ⇒ QA is still better then SA ??



# Conclusion

Kibble-Zurek argument was applied to the dynamics of SA and QA.

- model : pure and random Ising models in 1D
- quantities : density of kinks and residual energy
- analytic results have been confirmed by numerical tests

**QA has an advantage** at least in the 1D random model

**Randomness** induces the advantage of QA