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$N = 2$ Superconformal CP_n Model*

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ABSTRACT

We study the Feigin-Fuchs representation of the Kazama-Suzuki $N = 2$ coset model $CP_n = SU(n+1)/SU(n) \times U(1)$. The chiral algebra of this model is the $N = 2$ super W-algebra obtained from the quantum super Miura transformation associated with the Lie superalgebra $A(n, n-1)$. We construct screening operators, which commute with the W algebra, and investigate the null field structure of the model. We also study the chiral ring structure of the $N = 2$ CP_n model using the free field realization.

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1 Introduction

Two-dimensional $N = 2$ superconformal field theory ([1],[2],[3]) plays an important role in the compactification of superstrings on Calabi-Yau manifolds. In order to know physical properties of the theory compactified on such a non-trivial manifold, one should calculate correlation functions on Riemann surfaces, which has been done partially for $N = 2$ minimal models [4]. As another important aspect of the $N = 2$ theory, the twisted $N = 2$ superconformal algebra provides an example of topological conformal field theory [5], which has a remarkable agreement with two dimensional gravity coupled with minimal matter [6].

Kazama and Suzuki have shown that a large class of unitary conformal field theories with $N = 2$ superconformal symmetry can be realized as the coset model G/H, where G is a super Kac-Moody algebra and H its subalgebra and G/H is a Hermitian symmetric space [7]. The character of $N = 2$ coset model can be computed as the branching coefficient of an affine Lie algebra in the GKO construction [8]. However, it is a non-trivial problem to find the chiral algebra and to calculate correlation functions in this framework. A different approach has been proposed, which is based on Toda field theory over H in the case of G at level one[9]. In this model an $N = 2$ superconformal algebra is generated by vertex operator type currents, which can be regarded as screening operators in some bosonic coset models [10]. Unfortunately it is difficult to extend this approach to the case of generic level k. The geometrical approach based on $N = 2$ Landau-Ginzburg model [11] provides a powerful method to investigate the chiral ring structure of the model, which characterize the cohomological structure of the manifold G/H. However there is no apparent correspondence between the Landau-Ginzburg action and the G/H model at generic level k [12].

The Feigin-Fuchs (or free field) representation ([13],[14]) provides a fundamental tool for studying the representation of the chiral algebra and correlation functions for these models at generic level. The purpose of the present paper is to study $N = 2$ coset models using the free field realization. In a previous paper [15] we have studied the $N = 2$ CP_n coset model in view of the quantum Hamiltonian reduction of affine Lie superalgebra $A(n, n-1)^{(1)}$. We also have shown that the $N = 2$ CP_n model has an $N = 2$ super W-algebra structure through the supersymmetric Miura transformation ([16],[17]). The quantum Hamiltonian reduction of affine Lie algebra gives us a geometrical framework to understand the W algebra structure of the minimal coset models [18]. From the hamiltonian reduction of simply-laced affine Lie algebra $\hat{\mathbf{g}}$ at the level $k = p/q - h^\vee$ ($h^\vee$: dual Coxeter number of $\mathbf{g}$) we get (p, q) $W\mathbf{g}$ coset minimal models. For the W algebra with superconformal (or fermionic) symmetry, (affine) Lie superalgebra plays an essential role ([16],[17],[19],[20]).

The Feigin-Fuchs representation of the $W\mathbf{g}$ algebra has been studied extensively by many authors ([21]-[24]) since Zamolodchikov and Fateev have studied the $W_3 (= WA_2)$ algebra [21]. Fateev and Lukyanov have studied the Feigin-Fuchs representation of the $W\mathbf{g}$ algebras for $\mathbf{g} = A_n, B_n$ and D_n[22]. In these models $W\mathbf{g}$ algebras are obtained from the quantum Miura transformation. In the present paper we shall study the Feigin-Fuchs representation of $N = 2$ superconformal CP_n model by using screening operators which commute all the generators of the $N = 2$ super W-algebra.

This paper is organized as follows: In section 2, we study the free field representation of the $N = 2$ CP_n model based on the super Lax operator associated with the Lie superalgebra $A(n, n-1)$. In section 3, we construct screening operators and study the degenerate representation of the algebra. As an application, we investigate the chiral ring structure of the model.

2 $N = 2$ super W algebra

In this section we introduce the quantum Miura transformation for the $N = 2$ minimal CP_n model. This transformation can be derived from the Lie superalgebra $A(n, n-1)$

[25], which underlies the $N=2$ super W algebra.

2.1 Lie superalgebra $A(n,n-1)$

We start from the Lie superalgebra $\mathbf{g}=A(n,n-1)=sl(n+1,n)$ $(n\geq 1)$. The algebra $\mathbf{g}$ consists of $(2n+1)\times(2n+1)$ matrices $X=(x_{ij})$ satisfying

$$\mathrm{str}X\equiv\sum_{i=1}^{2n+1}(-1)^{i+1}x_{ii}=0, \tag{1}$$

and it has rank $2n$. The root system of the Lie superalgebra $\mathbf{g}$ is composed of even roots and odd roots, which correspond to commuting and anti-commuting generators, respectively. Let Δ^0 (Δ^1) be the set of even (odd) roots. For $A(n,n-1)$ one can choose a purely odd simple root system, in which the simple roots $\alpha_1,\ldots,\alpha_{2n}$ of $\mathbf{g}$ are written as

$$\alpha_{2i-1}=e_i-\delta_i,\quad \alpha_{2i}=\delta_i-e_{i+1}, \tag{2}$$

where e_i $(i=1,\ldots,n+1)$ and δ_i $(i=1,\ldots,n)$ are orthonormal basis of $\mathbf{R}^{n+1}$ and $\mathbf{R}^n$, respectively. $\mathbf{R}^{n+1}$ has a positive metric whereas $\mathbf{R}^n$ has a negative metric:

$$e_i\cdot e_j=\delta_{ij},\quad \delta_i\cdot\delta_j=-\delta_{ij}. \tag{3}$$

We denote the set of positive roots by Δ_+, which consists of

$$\alpha_i+\cdots+\alpha_j,\quad (i\leq j). \tag{4}$$

Here the even (odd) number sum of simple roots is even (odd). The set of negative roots Δ_- is defined as $(-1)\Delta_+$. The fundamental weights $\lambda_1,\ldots,\lambda_{2n}$ of $\mathbf{g}$ satisfy

$$\lambda_i\cdot\alpha_j=\delta_{ij}. \tag{5}$$

These are expressed in terms of α_i as

$$\begin{aligned}\lambda_{2i}&=\alpha_1+\alpha_3+\cdots+\alpha_{2i-1},\\ \lambda_{2i-1}&=\alpha_{2i}+\alpha_{2i+2}+\cdots+\alpha_{2n},\quad i=1,\ldots,n.\end{aligned} \tag{6}$$

The Lie superalgebra $A(n,n-1)$ contains an even subalgebra $A_n \oplus A_{n-1} \oplus C$, where $C = gl(1)$ is a center of $\mathbf{g}$ and generates a $U(1)$ subgroup. The simple roots of even subalgebra A_n are

$$\alpha_i^{(1)} = \alpha_{2i-1} + \alpha_{2i} = e_i - e_{i+1}, \quad i = 1, \ldots, n, \tag{7}$$

and those of A_{n-1}

$$\alpha_i^{(2)} = \alpha_{2i} + \alpha_{2i+1} = \delta_i - \delta_{i+1}, \quad i = 1, \ldots, n-1. \tag{8}$$

Note that the root system of A_{n-1} has negative metric. C is generated by ν:

$$\nu = \sum_{i=1}^{n} (\lambda_{2i} - \lambda_{2i-1}), \tag{9}$$

which has also a negative metric $\nu^2 = -n(n+1)$. Let $\Lambda_1^{(1)}, \ldots, \Lambda_n^{(1)}$ be the fundamental weights of A_n and $\Lambda_1^{(2)}, \ldots, \Lambda_{n-1}^{(2)}$ be those of A_{n-1}. Using λ_i's, we can construct the fundamental weights for even subalgebra of $\mathbf{g}$.

$$\begin{aligned} \Lambda_i^{(1)} &= \sum_{j=1}^{2i-1} (-1)^{j-1} \lambda_j + \frac{i}{n+1}\nu, \quad i = 1, \ldots, n, \\ \Lambda_i^{(2)} &= \sum_{j=1}^{2i} (-1)^{j} \lambda_j - \frac{i}{n}\nu, \quad i = 1, \cdots, n-1. \end{aligned} \tag{10}$$

Actually we can show that these vectors satisfy

$$\begin{aligned} \Lambda_i^{(1)} \cdot \Lambda_j^{(1)} &= \begin{cases} \frac{1}{n+1}(n+1-j)i, & \text{for } i \leq j, \\ \frac{1}{n+1}(n+1-i)j, & \text{for } i > j, \end{cases} \\ \Lambda_i^{(2)} \cdot \Lambda_j^{(2)} &= \begin{cases} -\frac{1}{n}(n-j)i, & \text{for } i \leq j, \\ -\frac{1}{n}(n-i)j, & \text{for } i > j, \end{cases} \end{aligned} \tag{11}$$

and that $\Lambda_i^{(1)}$, $\Lambda_j^{(2)}$ and ν are orthogonal. The highest weight of the representation of $A(n,n-1)$

$$\Lambda = \sum_{i=1}^{2n} m_i \lambda_i, \tag{12}$$

where m_i are non-negative integers, is decomposed into the sum of weights of even subalgebras:

$$\Lambda = \Lambda^{(1)} + \Lambda^{(2)} + Q\nu, \tag{13}$$

where

$$\Lambda^{(1)} = \sum_{i=1}^{n} N_i^{(1)} \Lambda_i^{(1)}, \qquad \Lambda^{(2)} = \sum_{i=1}^{n-1} N_i^{(2)} \Lambda_i^{(2)}, \tag{14}$$

and

$$\begin{aligned} N_i^{(1)} &= m_{2i} + m_{2i-1}, \qquad N_i^{(2)} = m_{2i} + m_{2i+1}, \\ Q &= \frac{1}{n(n+1)} \sum_{i=1}^{n} (i m_{2i} - (n+1-i) m_{2i-1}). \end{aligned} \tag{15}$$

These relations are crucial to establish the relation between the Lie superalgebra $A(n, n-1)$ and the $N = 2$ CP_n model.

2.2 Super Lax operator

In a previous paper we have discussed the quantum Hamiltonian reduction of the affine Lie superalgebra $A(n, n-1)^{(1)}$ and have shown that the $N = 1$ super Lax operator based on $A(n, n-1)$ characterize the chiral algebra structure of the $N = 2$ CP_n model [15]. Here we present a detailed analysis of the $N = 2$ super W algebra structure.

In the following we use the $N = 1$ superspace formalism. The super coordinate is denoted by $Z = (z, \theta)$ and the super derivative by $D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z}$. Let $\varphi^i(z)$ and $\chi^i(z)$ $(i = 1, \ldots, 2n)$ be free bosons and real fermions which satisfy

$$\varphi^i(z) \varphi^j(w) = -\delta_{ij} \log(z - w) + \cdots, \qquad \chi^i(z) \chi^j(w) = \frac{\delta_{ij}}{z - w} + \cdots. \tag{16}$$

We define scalar superfields $\Phi(Z)$ by $\varphi(z) + \mathrm{i}\theta\chi(z)$.

Let us consider the scalar super Lax operator $L(Z)$ ([16],[17]):

$$L(Z) =: (aD - \Theta_{2n+1}(Z))(aD - \Theta_{2n}(Z)) \cdots (aD - \Theta_1(Z)) :, \tag{17}$$

where

$$\Theta_i(Z) = (-1)^{i-1} h_i \cdot D\Phi(Z), \tag{18}$$

$a = -\mathrm{i}\alpha_+$, $h_i = \lambda_i - \lambda_{i-1}$ $(i = 1, \ldots, 2n+1)$, and $\lambda_0 = \lambda_{2n+1} = 0$. The symbol : : means normal ordering. Expanding $L(Z)$ in powers of aD, we get the generators of $N = 2$ super

W-algebra, $W_{k/2}(Z)$ $(k = 0, \ldots, 2n+1)$:

$$L(Z) = \sum_{k=0}^{2n+1} W_{\frac{k}{2}}(Z)(aD)^{2n+1-k}. \tag{19}$$

We show first few examples of the currents: Obviously $W_0(Z)$ is equal to 1. $W_{\frac{1}{2}}(Z)$ is shown to be $\sum_{i=1}^{2n+1} h_i \cdot D\Phi$, which vanishes. The next two currents are those of $N = 2$ superconformal algebra:

$$\begin{aligned} W_1(Z) &= \sum_{i=1}^{2n}(-\lambda_{i+1} \cdot D\Phi\lambda_i \cdot D\Phi + a(-1)^i\lambda_i \cdot D^2\Phi), \\ W_{\frac{3}{2}}(Z) &= a\Big\{\sum_{i=1}^{n}(-\lambda_{2i-1} \cdot D\Phi\alpha_{2i-1} \cdot D^2\Phi - a\lambda_{2i-1} \cdot D^3\Phi)\Big\}. \end{aligned} \tag{20}$$

In the component formalism they are expressed as

$$W_1(Z) = J(z) + \mathrm{i}\theta(G^+(z) + G^-(z)), \tag{21}$$

$$\frac{1}{a}W_{\frac{3}{2}}(Z) = \mathrm{i}G^-(z) + \theta(T(z) + \frac{1}{2}\partial J(z)), \tag{22}$$

where

$$\begin{aligned} J(z) &= \sum_{i=1}^{n}\lambda_{2i} \cdot \chi\alpha_{2i} \cdot \chi - \mathrm{i}\alpha_+\nu \cdot \partial\varphi, \\ G^+(z) &= \sum_{i=1}^{n}(\alpha_{2i} \cdot \partial\varphi\lambda_{2i} \cdot \chi - \mathrm{i}\alpha_+\lambda_{2i} \cdot \partial\chi), \\ G^-(z) &= -\sum_{i=1}^{n}(\alpha_{2i-1} \cdot \partial\varphi\lambda_{2i-1} \cdot \chi - \mathrm{i}\alpha_+\lambda_{2i-1} \cdot \partial\chi), \\ T(z) &= -\frac{1}{2}(\partial\varphi)^2 + \mathrm{i}\alpha_+\mu \cdot \partial^2\varphi - \frac{1}{2}\sum_{i=1}^{2n}\chi^i\partial\chi^i, \end{aligned} \tag{23}$$

and

$$\mu = \frac{1}{2}\sum_{i=1}^{2n}\lambda_i. \tag{24}$$

Formulas (20) imply that the apparent $N = 1$ superconformal symmetry which we use in order to introduce the superspace, is generated by $(G^+ + G^-)/\sqrt{2}$. We can show easily that these currents satisfy the operator product expansion of the N=2 superconformal algebra:

$$
\begin{aligned}
T(z)T(w) &= \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots, \\
T(z)J(w) &= \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w} + \cdots, \\
T(z)G^{\pm}(w) &= \frac{\frac{3}{2}G^{\pm}(w)}{(z-w)^2} + \frac{\partial G^{\pm}(w)}{z-w} + \cdots, \\
J(z)J(w) &= \frac{c/3}{(z-w)^2} + \cdots, \\
J(z)G^{\pm}(w) &= \frac{\pm G^{\pm}(w)}{z-w} + \cdots, \\
G^{+}(z)G^{-}(w) &= \frac{c/3}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{T(w)+\frac{1}{2}\partial J(w)}{z-w} + \cdots, \qquad (25)
\end{aligned}
$$

with the central charge c:

$$c = 3n(1-(n+1)\alpha_+^2). \qquad (26)$$

It is convenient to introduce a symplectic basis $\{\alpha_{2i}, \lambda_{2i}\}$ $(i = 1, \ldots, n)$ for the root space in order to express the $N = 2$ generators in terms of n complex bosons and n complex fermions defined as $\varphi_i(z) = \lambda_{2i} \cdot \varphi(z)$, $\bar{\varphi}_i(z) = \alpha_{2i} \cdot \varphi(z)$, $\chi_i(z) = \lambda_{2i} \cdot \chi(z)$ and $\bar{\chi}_i(z) = \alpha_{2i} \cdot \chi(z)$. Using these fields, generators of the $N = 2$ superconformal algebra are expressed as

$$
\begin{aligned}
J(z) &= \sum_{i=1}^{n} \chi_i\bar{\chi}_i - \mathrm{i}\alpha_+ \sum_{i=1}^{n} (\partial\varphi_i - i\partial\bar{\varphi}_i), \\
G^{+}(z) &= \sum_{i=1}^{n} (\partial\bar{\varphi}_i\chi_i - \mathrm{i}\alpha_+\partial\chi_i), \qquad G^{-}(z) = -\sum_{i=1}^{n} (\partial\varphi_i\bar{\chi}_i - \mathrm{i}\alpha_+ i\partial\bar{\chi}_i), \\
T(z) &= \sum_{i=1}^{n} \{-\partial\varphi_i\partial\bar{\varphi}_i + \frac{\mathrm{i}\alpha_+}{2}\partial^2(\varphi_i + i\bar{\varphi}_i) - \frac{1}{2}(\chi_i\partial\bar{\chi}_i + \bar{\chi}_i\partial\chi_i)\}. \qquad (27)
\end{aligned}
$$

Note that for $n = 1$ we get the usual free field realization of $N = 2$ minimal model ([2],[3]). This kind of the free field realization has also been obtained in ref. [26], in a different approach.

Next we proceed to other W currents with spins larger than or equal to two. A general formula of the current is given by

$$W_{\frac{k}{2}}(Z)=\sum_{1\le l_1<\cdots<l_k\le 2n+1}(-1)^{\sum_{i=1}^{\left[\frac{k}{2}\right]}(l_{2i}-l_{2i-1}-1)}(aD-\Theta_{l_k})\cdots(aD-\Theta_{l_2})(-\Theta_{l_1}). \qquad (28)$$

After a suitable redefinition by adding some differential polynomials of the W currents with lower dimensions, they form an $N=2$ supermultiplet ($W_i^0(z)$, $W_i^+(z)$,$W_i^-(z)$, $W_i^2(z)$) $(i=1,\cdots,n)$ with conformal weights $(i,i+1/2,i+1/2,i+1)$ and $U(1)$ charges $(0,1,-1,0)$, respectively.

At the classical level the commutation relation for the W currents is interpreted as the Gelfand-Dickii algebra constructed from an appropriate affine Lie (super)algebra[27]. Recently it has been shown that the classical $N=2$ W-algebra can be obtained from affine Lie superalgebra $A(n,n)^{(1)}$ [28]. The relation between the present construction based on the Lie superalgebra $A(n,n-1)$ and that based on the affine Lie superalgebra $A(n,n)^{(1)}$ is not clear. Slightly different formulations of the classical $N=2$ super W-algebra have been proposed in refs. [29] and [30].

In order to proceed to the quantum case, we should generalize Lukyanov's approach [23] to the Lie superalgebra case. In the present paper, we do not study a detailed algebraic structure of the $N=2$ $W-$algebra. In the case of $n=2$ there are two $N=2$ supermultiplet $(J(z),G^+(z),G^-(z),T(z))$ and $(W_2^0(z),W_2^+(z),W_2^-(z),W_2^2(z))$. From the super Miura transformation (17), the $N=2$ supermultiplet containing $W_2^0(z)$ is expressed as:

$$\begin{aligned} W_2(Z) &= U^{(1)}U^{(2)}+aD\bar{\Phi}_2V^{(1)}-aD^2\bar{\Phi}_2U^{(2)}-aD^3\bar{\Phi}_2, \qquad (29)\\ W_{5/2}(Z) &= U^{(1)}\bar{V}^{(2)}+U^{(2)}\bar{V}^{(1)}+a^2D^2\bar{V}^{(1)}-aD\bar{\Phi}_2DV^{(1)}-aU^{(2)}D^3\bar{\Phi}_2-a^3D^5\bar{\Phi}_2, \end{aligned}$$

where $\Phi_i\equiv\lambda_{2i}\cdot\Phi$, $\bar{\Phi}_i\equiv\alpha_{2i}\cdot\Phi$,

$$\begin{aligned} U^{(i)}(Z) &= D\Phi_{2i}D\bar{\Phi}_{2i}-aD^2(\Phi_{2i}-\bar{\Phi}_{2i}),\\ \bar{V}^{(i)}(Z) &= -D^2\Phi_{2i}D\bar{\Phi}_{2i}-aD^3\bar{\Phi}_{2i}, \qquad V^{(i)}(Z)=DU^{(i)}-\bar{V}^{(i)}, \qquad (30) \end{aligned}$$

for $i=1,2$. One can show that these form an $N=2$ supermultiplet by an explicit calculation. Moreover one finds that $W_2^0(z)W_2^0(w)$ closes. Other operator product expansions and the generalization to the case of arbitrary n is presently under investigation and will be reported elsewhere[1] .

3 Degenerate representation

In this section we study the complete degenerate representation using the free field realization of the N=2 CP_n model. Let us consider a vertex operator of the form

$$V_\Lambda(z) = \mathrm{e}^{-\mathrm{i}\alpha_+\Lambda\cdot\varphi(z)}, \tag{31}$$

which has a conformal weight Δ_Λ and a U(1) charge q:

$$\Delta_\Lambda = \frac{\alpha_+^2}{2}\Lambda\cdot(\Lambda+2\mu), \tag{32}$$

$$q = -\alpha_+^2\Lambda\cdot\nu. \tag{33}$$

Let $\{V_\Lambda^0(z), V_\Lambda^+(z), V_\Lambda^-(z), V_\Lambda^2(z)\}$ be a N=2 supermultiplet from $V_\Lambda(z)$:

$$\begin{aligned}
V_\Lambda^0(z) &= \mathrm{e}^{-\mathrm{i}\alpha_+\Lambda\cdot\varphi(z)},\\
V_\Lambda^+(z) &= \mathrm{i}\alpha_+\Lambda_0\cdot\chi\mathrm{e}^{-\mathrm{i}\alpha_+\Lambda\cdot\varphi(z)}, \qquad V_\Lambda^-(z) = \mathrm{i}\alpha_+\Lambda_1\cdot\chi\mathrm{e}^{-\mathrm{i}\alpha_+\Lambda\cdot\varphi(z)},\\
V_\Lambda^2(z) &= \Big\{\alpha_+^2\Lambda_0\cdot\chi\Lambda_1\cdot\chi + \frac{1}{2}\mathrm{i}\alpha_+(\Lambda_1-\Lambda_0)\cdot\partial\varphi\Big\}\mathrm{e}^{-\mathrm{i}\alpha_+\Lambda\cdot\varphi(z)},
\end{aligned} \tag{34}$$

where we define Λ_0 and Λ_1 for $\Lambda = \sum_{i=1}^{2n} m_i\lambda_i$ as

$$\Lambda_0 = \sum_{i=1}^{n} m_{2i}\lambda_{2i}, \qquad \Lambda_1 = \sum_{i=1}^{n} m_{2i-1}\lambda_{2i-1}. \tag{35}$$

[1] After the completion of the present work, we have learned that the closure of $N=2$ $WA(2,1)$ has been established in [31]. Another construction of the $N=2$ $WA(2,1)$-algebra is proposed, based on the OPE method, in [32].

3.1 W charge and global symmetry

Here we compute the W charges $w_{k/2}(\Lambda)$ $(k=0,\cdots 2n+1)$ for a primary field $V_\Lambda(Z)$. They are defined by the operator product expansions with $W_{k/2}(Z)$:

$$\begin{aligned} W_j(Z_1)V_\Lambda(Z_2) &= \frac{w_j(\Lambda)V_\Lambda(Z_2)}{Z_{12}^j} + \text{less singular terms}, \\ W_{j-1/2}(Z_1)V_\Lambda(Z_2) &= \frac{\theta_{12}w_{j-1/2}(\Lambda)V_\Lambda(Z_2)}{Z_{12}^j} + \text{less singular terms}, \end{aligned} \tag{36}$$

where $Z_{12}=z_1-z_2-\theta_1\theta_2$ and $\theta_{12}=\theta_1-\theta_2$. Inserting this expression into eqs. (19) and using

$$(aD_1-\Theta_j(Z_1))V_\Lambda(Z_2)=(aD_1+\frac{a\theta_{12}(-1)^{j-1}h_j\cdot\Lambda}{Z_{12}})V_\Lambda(Z_2)+\cdots, \tag{37}$$

for (17), we get

$$\begin{aligned} &\sum_{j=0}^{n}(\frac{w_j(\Lambda)}{Z_{12}^j}(aD_1)^{2n+1-2k}+\frac{w_{j-1/2}(\Lambda)\theta_{12}}{Z_{12}^j}(aD_1)^{2n-2k}) \\ &= (aD_1+\frac{\theta_{12}h_{2n+1}\cdot\Lambda}{Z_{12}})\cdots(aD_1+\frac{\theta_{12}h_1\cdot\Lambda}{Z_{12}}). \end{aligned} \tag{38}$$

Applying this equation to monomials Z_{12}^j and $\theta_{12}Z_{12}^j$ $(j=0,\ldots,2n+1)$, we get the system of linear equations for the W charges:

$$\begin{aligned} \sum_{k=n-j+1}^{n}\frac{(j+1)!}{(j+1-n+k)!}a^{2n-2k}w_{k+\frac{1}{2}}(\Lambda) &= a^{2n+1}\prod_{m=0}^{n}(j-m+1+h_{2m+1}\cdot\Lambda) \\ &\quad -(j+1)a^{2n+1}\prod_{m=1}^{n}(j-m+1-h_{2m}\cdot\Lambda), \\ \sum_{k=n-j}^{n}a^{2n+1-2k}\frac{j!}{(j-n+k)!}w_k(\Lambda) &= a^{2n+1}\prod_{m=1}^{n}(j-m+1-h_{2m}\cdot\Lambda). \end{aligned} \tag{39}$$

This means that the W charges $w_{k/2}$ are invariant under the following discrete transformations:

$$\begin{aligned} -m+h_{2m}\cdot\Lambda &= -m'+h_{2m'}\cdot\Lambda, \\ -m+h_{2m+1}\cdot\Lambda &= -m''+h_{2m''+1}\cdot\Lambda, \end{aligned} \tag{40}$$

where m' and m'' are obtained from the numbers $m = (1, \ldots, n)$ by some permutations. From the relation (10) we get

$$\begin{aligned} h_{2m} &= \Lambda_m^{(2)} - \Lambda_{m-1}^{(2)} + \frac{\nu}{n}, \\ h_{2m+1} &= \Lambda_{m+1}^{(1)} - \Lambda_m^{(1)} - \frac{\nu}{n+1}. \end{aligned} \tag{41}$$

Therefore the discrete symmetry mentioned above is nothing but the outer automorphism of even subalgebras A_n and A_{n-1}, which is that of the Lie superalgebra $A(n, n-1)$. This symmetry determines the identification of vertex operators which represent a primary field of the $N = 2$ W algebra. From this discrete symmetry we may calculate the fusion rules for the $N = 2$ CP_n models, which will be discussed elsewhere.

3.2 Screening operators

In order to study the representation of the algebra by using the free field realization, we must introduce screening operator which commutes with the generators of the chiral algebra of the model. In the present case the screening operator $S(z)$ is defined as

$$W_i^a(z)S(w) = \frac{\partial}{\partial w}\left(\sum_{j\geq 0} \frac{S_j'(w)}{(z-w)^j}\right) + \ldots, \qquad a = 0, +, -, 2, \quad i = 1, \ldots, n. \tag{42}$$

where $S_j'(w)$ are local operators. Although we leave the closure of the $N = 2$ super W-algebra for arbitrary n for the future work, we can calculate the screening operator $S(Z) = S_0(z) + \theta S(z)$ which commutes with the super Lax operator $L(Z)$:

$$L(Z_1)S(Z_2) = \sum_{k=0}^{2n+1} D_2(X_k)(aD_1)^{2n+1-k} + \cdots, \tag{43}$$

where $S_0(z)$ is a superpartner of $S(z)$ and X_k take the form of operator product expansions. Using the super Miura transformation (19) and comparing both sides of of eq. (43) with respect to θ_2, one finds (43) is equivalent to (42).

We note that the $N = 2$ minimal models have three screening operators[2], [3]:

$$\begin{aligned} S_1^+(z) &= \chi(z)\mathrm{e}^{\mathrm{i}\alpha_-\varphi(z)}, \qquad S_2^+(z) = \bar{\chi}(z)\mathrm{e}^{\mathrm{i}\alpha_-\bar{\varphi}(z)}, \\ S^-(z) &= (\alpha_+^2\chi\bar{\chi}(z) + \frac{1}{2}\mathrm{i}\alpha_+(\partial\bar{\varphi} - \partial\varphi))\mathrm{e}^{\mathrm{i}\alpha_+(\varphi+\bar{\varphi})(z)}, \end{aligned} \tag{44}$$

where $\alpha_- = -1/\alpha_+$. The first two are fermionic screening operators which come from the hamiltonian reduction of $A(1,0)^{(1)}$ ([33],[15]), which characterize the Lie superalgebraic structure of the model. The remaining bosonic screening operator is used to determine the $A_1^{(1)}$ structure of the model since the $N = 2$ minimal model is obtained from the marginal deformation of $SL(2)$ Wess-Zumino-Novikov-Witten model [34]. In the coset realization of the $N = 2$ minimal model $CP_1 = SU(2)/U(1)$ the screening operator $S^-(z)$ is thought to characterize the structure of $SU(2)$.

We will show that this observation can be generalized to arbitrary n. Actually we can find three types screening operators for the $N = 2$ coset model $SU(n+1)/SU(n) \times U(1)$. One is fermionic screening operator which comes from the Hamiltonian reduction of $A(n, n-1)^{(1)}$ and take the form [15]:

$$S_j(z) = \alpha_j \cdot \chi \mathrm{e}^{\mathrm{i}\alpha_- \alpha_j \cdot \varphi(z)}, \qquad j = 1, \ldots, 2n, \tag{45}$$

where $\alpha_- = -1/\alpha_+$. Other two types are used for the characterization of $SU(n+1)$ and $SU(n)$ structure of the model. They are expressed as

$$S_i^1(z) = \{\alpha_+^2 \alpha_{2i-1} \cdot \chi \alpha_{2i} \cdot \chi + \frac{1}{2}\mathrm{i}\alpha_+(\alpha_{2i} - \alpha_{2i-1}) \cdot \partial\varphi\} \mathrm{e}^{\mathrm{i}\alpha_+(\alpha_{2i}+\alpha_{2i-1})\cdot\varphi(z)}, \tag{46}$$

for $i = 1, \ldots, n$ and

$$S_i^2(z) = \{\alpha_+^2 \alpha_{2i+1} \cdot \chi \alpha_{2i} \cdot \chi + \frac{1}{2}\mathrm{i}\alpha_+(\alpha_{2i} - \alpha_{2i+1}) \cdot \partial\varphi\} \mathrm{e}^{-\mathrm{i}\alpha_+(\alpha_{2i}+\alpha_{2i+1})\cdot\varphi(z)}, \tag{47}$$

for $i = 1, \ldots, n-1$. We can show these operators commute with the super Lax operator $L(Z)$. For the fermionic screening operator $S_j(z)$, which has a supersymmetric form $\exp(\mathrm{i}\alpha_- \alpha_j \cdot \Phi(Z))$, it is easily shown that

$$(aD_1 - \Theta_{j+1}(Z_1)(aD_1 - \Theta_j(Z_1))\mathrm{e}^{\mathrm{i}\alpha_- \alpha_j \cdot \Phi(Z_2)} = D_2(\frac{\theta_{12} \mathrm{e}^{\mathrm{i}\alpha_- \alpha_j \cdot \Phi(Z_2)}}{Z_{12}}) + \cdots. \tag{48}$$

Other $(aD_1 - \Theta_{j'}(Z_1))$ does not produce the singular part. Therefore $S_j(z)$ commutes with the whole generators of the $N = 2$ super W algebra. For the bosonic screening operator $S_i^1(z)$, which has a supersymmetric form $S_i^1(Z) = (\alpha_{2i} - \alpha_{2i-1}) \cdot D\Phi V_{-\alpha_i^{(1)}}(Z)$, we find

$$\begin{aligned}
& (aD_1 - \Theta_{2i+1}(Z_1))(aD_1 - \Theta_{2i}(Z_1))(aD_1 - \Theta_{2i-1}(Z_1))S_i^1(Z_2) \\
= \; & D_2(\frac{\theta_{12}(-a\Lambda \cdot D\Phi)V_{-\alpha_i^{(1)}}}{Z_{12}})aD_1 \\
& +aD_2\Big\{\frac{\alpha_{2i} \cdot D\Phi}{Z_{12}}V_{-\alpha_i^{(1)}} + \frac{\theta_{12}}{Z_{12}}[(\alpha_{2i} - \alpha_{2i-1}) \cdot D^2\Phi + 2a\alpha_{2i-1} \cdot D\Phi\alpha_{2i} \cdot D\Phi \\
& \quad -\alpha_{2i} \cdot D\Phi h_{2i-1} \cdot D\Phi + h_{2i+1} \cdot D\Phi\alpha_{2i-1} \cdot D\Phi]V_{-\alpha_i^{(1)}}\Big\} + \cdots . \qquad (49)
\end{aligned}$$

Hence the operator product expansion between the super Lax operator $L(Z)$ and $S_i^1(Z)$ takes the form of (43). In a similar way we can show that S_i^2 commutes with the generators of the $N = 2$ super W-algebra.

3.3 Null field construction

Let $\Phi_\Lambda(z)$ be a primary fields with the W charges $w_{k/2}(\Lambda)$ given by the formula (39). $[\Phi_\Lambda]$ is a Verma module with the highest weight Λ built on $\Phi_\Lambda(z)$. In the Feigin-Fuchs representation we treat the Fock module $[V_\Lambda]$ built on a vertex operator $V_\Lambda(z)$. $[V_{-2\mu-\Lambda}]$ is a dual Fock space to $[V_\Lambda]$. Due to $Z_n \times Z_{n-1}$ symmetry, we identify $[V_\Lambda]$ with $[V_{\Lambda^*}]$, where Λ^* is obtained by the permutation of $(\Lambda_1^{(1)}, \ldots, \Lambda_n^{(1)})$ or $(\Lambda_1^{(2)}, \ldots, \Lambda_{n-1}^{(2)})$ in Λ. In the following we consider the case $\alpha_+^2 = 1/(k + n + 1)$, which gives the central charge

$$c = \frac{3kn}{k + n + 1}, \qquad (50)$$

of the $N = 2$ CP_n coset model. In order to study the degenerate representation of the algebra, we must find a null field $\chi_\Lambda(z)$ in $[V_\Lambda]$, which can be constructed explicitly by applying screening operators $S(z)$ to an appropriate vertex operator $V_{\Lambda'}(z)$:

$$\chi_\Lambda(z) = \int du_1 \cdots du_r S(u_1) \cdots S(u_r) V_{\Lambda'}(z), \qquad (51)$$

where contours of the integration are taken as, for example, ref. [35]. If the above integral exist non-zero, $\chi_\Lambda(z)$ becomes a null field and $[\chi_\Lambda]$ generates a submodule in $[V_\Lambda]$.

In the present case we can construct three types of null fields in the Fock module $[V_\Lambda]$ from screening operators $S_j(z)$, $S_i^1(z)$ and $S_i^2(z)$. For the fermionic screening operator $S_j(z)$ a null field is expressed as

$$\begin{aligned}\chi_\Lambda(z) &= \int du S_j(u) V_{\Lambda'}(z), \\ &= \int du (u-z)^{\alpha_j\cdot\Lambda'} : \chi(u) \mathrm{e}^{\mathrm{i}\alpha_-\alpha_j\cdot\varphi(u)-\mathrm{i}\alpha_+\Lambda'\cdot\varphi(z)},\end{aligned} \tag{52}$$

where $\Lambda' = -2\mu - \Lambda + \alpha_j/\alpha_+^2$. The non-zero existence of the above integral requires

$$-\alpha_j \cdot \Lambda - 1 = -N_j, \tag{53}$$

where N_j is a positive integer. In this case $\chi_\Lambda(z)$ is a null field in $[V_{-2\mu-\Lambda}]$ at level N_j. From eq. (53) we get

$$\Lambda = \sum_{i=1}^{2n} (N_i - 1)\lambda_i. \tag{54}$$

For the bosonic screening operator $S_i^1(z)$, we get results similar to the WA_n algebra [22]. The null field is expressed as

$$\chi_\Lambda(z) = \int du_1 \cdots du_{r_i} \prod_{j=1}^{r_i} S_i^1(u_j) V_{\Lambda'}(z), \tag{55}$$

with $\Lambda' = -2\mu - \Lambda + r\alpha_i^{(1)}$. The non zero existence of $\chi_\Lambda(z)$ requires

$$\alpha_+^2 r_i(r_i - 1) - r_i\alpha_+^2\alpha_i^{(1)} \cdot \Lambda' = -r_i s_i, \tag{56}$$

where s_i is a positive integer. In this case $\chi_\Lambda(z)$ is a null field in $[V_{-2\mu-\Lambda}]$ at level $r_i s_i$. Hence we get

$$\Lambda = \sum_{i=1}^{n} \{-r_i + \frac{s_i}{\alpha_+^2}\}\Lambda_i^{(1)}. \tag{57}$$

Similarly, for screening operators $S_i^2(z)$, the highest weight takes the form

$$\Lambda = \sum_{i=1}^{n-1} \{-r_i' - \frac{s_i'}{\alpha_+^2}\}\Lambda_i^{(2)}, \tag{58}$$

for positive integers r_i' and s_i'. Next we consider the $U(1)$ current. Let us bosonize the $U(1)$ current:

$$J(z) = \mathrm{i}\sqrt{\frac{c}{3}}\partial\Phi, \tag{59}$$

where $\Phi(z)$ is a free boson compactified on a circle with radius

$$R = \sqrt{\frac{c}{3}} = \sqrt{\frac{k+n+1}{kn}}. \tag{60}$$

A primary field is expressed by the vertex operator

$$\exp(\frac{\mathrm{i}m\Phi}{\sqrt{kn(k+n+1)}}), \quad m : \text{integer}, \tag{61}$$

which has a $U(1)$ charge

$$q = \frac{m}{k+n+1}. \tag{62}$$

Combining these results, a primary field $V_\Lambda(z)$ should have the weight

$$\Lambda = \sum_{i=1}^{n} l_i^{(1)}\Lambda_i^{(1)} + \sum_{i=1}^{n-1} l_i^{(2)}\Lambda_i^{(2)} + Q\nu, \tag{63}$$

with

$$\begin{aligned} l_i^{(1)} &= -r_i + s_i(k+n+1), \quad i = 1, \ldots, n, \\ l_i^{(2)} &= -r_i' - s_i'(k+n+1), \quad i = 1, \ldots, n-1, \\ Q &= \frac{-m}{n(n+1)}. \end{aligned} \tag{64}$$

Finally we consider the fermion sector which represents affine Lie algebra $SO(2n)$ at level one, since the $N = 2$ G/H model is equivalent to the bosonic one $G \times SO(2\mathrm{dim}(G/H))/H$. It is convenient to bosonize complex fermions $\chi_i(z)$ and $\bar{\chi}_i(z)$ such as

$$\chi_i(z) = \mathrm{e}^{\mathrm{i}\phi_i(z)}, \qquad \bar{\chi}_i(z) = \mathrm{e}^{-\mathrm{i}\phi_i(z)}, \qquad i = 1, \ldots, n. \tag{65}$$

The primary field takes the form:

$$\mathrm{e}^{\mathrm{i}\tilde{\Lambda}\cdot\phi}, \tag{66}$$

where $\tilde{\Lambda}$ is a weight of $SO(2n)$. From (31) and (66), we get the primary field of the $N = 2$ CP_n minimal model:

$$V_{\Lambda,\tilde{\Lambda}}(z) = \mathrm{e}^{\mathrm{i}\tilde{\Lambda}\cdot\phi(z)}\mathrm{e}^{-\mathrm{i}\alpha_+\Lambda\cdot\varphi(z)}. \tag{67}$$

Actually the conformal dimension $\Delta_{\Lambda,\tilde{\Lambda}}$ and the $U(1)$ charge q of $V_{\Lambda,\tilde{\Lambda}}(z)$ are given as

$$\begin{aligned}\Delta &= \frac{\Lambda^{(1)}(\Lambda^{(1)}+2\rho^{(1)})+\Lambda^{(2)}(\Lambda^{(2)}+2\rho^{(2)})}{2(k+n+1)} - \frac{m^2}{2n(n+1)(k+n+1)} + \frac{1}{2}\tilde{\Lambda}^2, &(68)\\ q &= \frac{m}{k+n+1} + \sum_{i=1}^{n}\tilde{\Lambda}_i, &(69)\end{aligned}$$

where we use the following formula

$$\mu = \rho^{(1)} + \rho^{(2)}, \tag{70}$$

and $\rho^{(1)}$ and $\rho^{(2)}$ are the Weyl vectors of the even subalgebras A_n and A_{n-1}, respectively:

$$\rho^{(1)} = \sum_{i=1}^{n}\Lambda_i^{(1)}, \qquad \rho^{(2)} = \sum_{i=1}^{n-1}\Lambda_i^{(2)}. \tag{71}$$

These formula shows that the present model is nothing but the $N=2$ CP_n coset model[7].

3.4 Chiral ring structure

As an application of the present free field realization, we study the chiral ring structure of the $N=2$ CP_n model. First we review that of the $N=2$ G/H coset model [12]. For the integrable highest weight Λ of the affine Lie algebra $\hat{\mathbf{g}}$ of a Lie group G and an element w of the Weyl group $W(G)$ of the Lie algebra $\mathbf{g}$ of G, a chiral primary field C_w^Λ is expressed as

$$C_w^\Lambda = \psi^w G^\Lambda_{w^{-1}(\Lambda)}, \tag{72}$$

where

$$\psi^w = \prod_{\alpha\in\Delta_+(G/H)\cap w^{-1}(\Delta_-)} \psi^\alpha, \tag{73}$$

and G^Λ_λ is a primary fields with weight λ in the representation of $\hat{\mathbf{g}}$ with the highest weight Λ. Here $(\psi^\alpha, \psi^{-\alpha})$ $(\alpha \in \Delta_+(G/H))$ are complex fermions which represents the affine Lie algebra $SO(2\mathrm{dim}(G/H))$ at level one.

In the case of the $N=2$ CP_n coset model, $\Delta_+(G/H) = \{\alpha_1 + \cdots + \alpha_i, \;\; i=1,\ldots,n\}$. We may identify χ_i and $\bar{\chi}_i$ as $\psi^{\alpha_1+\cdots+\alpha_i}$ and $\psi^{-(\alpha_1+\cdots+\alpha_i)}$, respectively. Let us consider the structure of the bosonic part which is characterized by the representation of $\hat{\mathbf{g}} = A_n^{(1)}$. Let $V_\Lambda(z)$ be a bosonic part of the chiral primary field, which satisfies

$$G^+(z)V_\Lambda(w) = \text{regular}. \tag{74}$$

This implies V_Λ^+ is equal to zero. From (34) we find $\Lambda_0 = 0$. Therefore Λ is expressed as $\sum_{i=1}^n m_{2i-1}\Lambda_{2i-1}$. From (15) Λ is given in terms of the weights of even subalgebra $(\Lambda^{(1)}, \Lambda^{(2)}, Q)$ such as

$$\begin{aligned} \Lambda^{(1)} &= \sum_{i=1}^{n} l_i \Lambda_i^{(1)}, \qquad \Lambda^{(2)} = \sum_{i=1}^{n-1} l_{i+1}\Lambda_i^{(2)}, \\ Q &= \frac{-1}{n(n+1)}\sum_{i=1}^{n}(n+1-i)l_i. \end{aligned} \tag{75}$$

Hence the chiral primary field can be characterized by the weight of the even subalgebra A_n. Note that the $U(1)$ charge take the form

$$q = \frac{(n+1)\Lambda_1^{(1)} \cdot \Lambda^{(1)}}{k+n+1}. \tag{76}$$

Hence the conformal weight Δ_Λ and $U(1)$ charge q are invariant under the Weyl reflection corresponding to the root $\alpha_i^{(1)} + \cdots + \alpha_j^{(1)}$ $(2 \le i \le j \le n)$. Combining this and the discrete automorphism Z_{n+1} of $A_n^{(1)}$, we find that the number of chiral primary fields of $N = 2$ CP_n model is given by

$$\frac{1}{\mid Z_{n+1} \mid} N_{A_n}^k \frac{\mid W(A_n) \mid}{\mid W(A_{n-1}) \mid}, \tag{77}$$

where $N_{A_n}^k$ is the number of the integrable highest weights of $A_n^{(1)}$ at level k. This agree with the result of [12]. The free field realization provides an explicit construction of the chiral primary fields in a rather simplified manner. This would be useful to investigate correlation functions among chiral primary fields on Riemann surfaces.

4 Conclusions and discussion

In the present paper we have studied the Feigin-Fuchs representation of the $N = 2$ superconformal CP_n model. We have constructed the fermionic and bosonic screening operators which commute with the $N = 2$ super W algebra. The fermionic screening operators characterize the Lie superalgebraic structure of the $N = 2$ CP_n model. On the other hand, the bosonic ones characterize the coset structure of the model. Using these operators we have investigated the null field structure of the $N = 2$ CP_n model.

There still remain a few problems to be solved, such as the computation of correlation functions, characters, fusion rules etc. Concerning the character, it is a fundamental technique to introduce Felder's BRST cohomological structure among the space of Fock modules[37]. For the $N = 2$ CP_n model we can take fermionic screening operators as the BRST operators

$$Q_j = \int dz S_j(z), \tag{78}$$

as in the case of the $N = 2$ minimal models [3],[38]. These operators satisfy

$$\begin{aligned} Q_i^2 &= 0, \\ Q_i Q_j + \epsilon_{ij} Q_j Q_i &= 0, \end{aligned} \tag{79}$$

where $\epsilon_{ij} = \exp(\pm i\pi\alpha_+^{-2})$ for $j = i \pm 1$ and 1 otherwise. Q_i defines a map between $[V_{-2\mu-\Lambda+\alpha_i/\alpha_+^2}]$ to $[V_{-2\mu-\Lambda}]$. If $\Delta_\Lambda < \Delta_{\Lambda-\alpha_i/\alpha_+^2}$, for the state v in KerQ_i, there is a state v' such that

$$v = Q_i v'. \tag{80}$$

Therefore Q_i defines a BRST cohomology in the Fock spaces. Using this we can compute the character formula for the $N = 2$ CP_n model. For $n = 2$ we get a result similar to that obtained from the branching coefficient of the affine Lie algebra[36]. However the precise relation between the two results is not clear. A detailed analysis will be presented elsewhere.

Note that in the Kazama-Suzuki models there is a kind of duality relation such as

- CP_n model at level $1 \equiv CP_1$ model at level n.

- CP_n model at level $k \equiv$ a Grassmannian coset model $SU(n+k)/SU(n)\times SU(k)\times U(1)$ at level 1.

The present Feigin-Fuchs representation provides a new kind of free field representation for these coset models. In particular this means that the A_n type $N=2$ minimal series might have an extended algebraic structure, although at first sight this is not manifest by looking at the character for these models.

It is worth noting that the present $N=2$ W-algebra structure also appears in the topological $SL(n,R)$ gravity[39]. Hence it is natural to consider the topological $SL(n,R)$ gravity coupled with the twisted $N=2$ CP_n model as a generalization of [6]. We expect that this model would describe the non-perturbative aspect of the two-dimensional W gravity coupled with the W minimal matter.

It is important to generalize the present Feigin-Fuchs representation to other types of Kazama-Suzuki models. A class of such free field realizations has been found in a non-Lie algebraic approach in ref. [26]. A geometrical viewpoint for the construction of the free field realization, is clearly desirable to understand the integrability property and the chiral algebra structure of these models.

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