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Feigin-Fuchs Representations of Arbitrary Affine Lie Algebras

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ABSTRACT

We develop a systematic method to obtain the Feigin-Fuchs representations for arbitrary affine Lie algebras from a geometrical view point. Choosing canonical coordinates for the flag manifolds associated with the Lie algebras, we get the general formulas for the Kac-Moody currents. In particular we use this method to construct explicitly the Feigin-Fuchs representations for the affine Lie algebras of $G_2^{(1)}$ type.

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Conformal field theory plays a fundamental role in the investigation of string theory and two-dimensional critical phenomena. The Feigin-Fuchs (or Coulomb gas) representation of conformal field theory gives a powerful tool to study the representations of the chiral algebras and to calculate the correlation functions on Riemann surfaces and their monodromy matrices, which reveal their quantum group structure. In particular the Wess-Zumino-Witten (WZW) models, whose chiral algebras are affine Lie algebras, are important because these models are considered to be "building blocks" of all rational conformal field theories through the coset construction.

Recently the Feigin-Fuchs representations of WZW models are extensively studied by many authors ([1],[2],[3],[4],[5],[6],[7],[8],[9],[10],[11]). Wakimoto proposed a representation of the affine Lie algebra $A_1^{(1)}$ at level 1 ([1]). Zamolodchikov generalized this representation to arbitrary level k ([2]). In this construction the Kac-Moody currents are realized by a pair of β-γ bosonic ghosts and a free boson coupled to the world sheet curvature. A generalization to the case of the affine Lie algebras $A_n^{(1)}$ was done by several authors ([4],[8],[10]). For other types of the affine Lie algebras, an explicit construction of the affine Lie algebra $C_2^{(1)}$ was given in ref. [5]. In ref. [11] the Feigin-Fuchs representations of the affine Lie algebras of $B_n^{(1)}$, $C_n^{(1)}$ and $D_n^{(1)}$ types were proposed, but one can check that the currents corresponding to the positive roots do not satisfy the operator product relations. Therefore it is necessary to construct the currents which satisfy the correct operator product expansions for these types of affine Lie algebras.

In this letter we propose a systematic derivation of the Feigin-Fuchs representations for general affine Lie algebras from the geometrical point of view. Our construction has a manifest invariance under the (outer-)automorphism of the Lie algebra, which is not manifest in the usual construction of the affine Lie algebras $A_n^{(1)}$.

Let G be a complex simple Lie group corresponding to a complex simple Lie algebra $\mathbf{g}$. G admits a Gauss decomposition $G = N_+ H N_-$, where $N_-(N_+)$ is a unipotent subgroup of G generated by the negative (positive) roots of $\mathbf{g}$ and H is generated by the Cartan subalgebra of $\mathbf{g}$. We shall denote the corresponding subalgebras by $\mathbf{n}_-$ ($\mathbf{n}_+$) and $\mathbf{h}$. Any element g of G is decomposed as follows:

$$g = \zeta \delta z, \tag{1}$$

where ζ (z) is a lower(upper)-triangular matrix with ones along the diagonal and δ is a diagonal matrix. The Chevalley generators E_α $(\alpha \in \Delta)$ and H^i $(i = 1, \ldots, r)$ of the Lie algebra $\mathbf{g}$, where Δ is the set of the roots of $\mathbf{g}$ and r is the rank of $\mathbf{g}$, satisfy the following commutation relations:

$$\begin{aligned} [E_\alpha, E_\beta] &= N_{\alpha,\beta} E_{\alpha+\beta}, \\ [E_\alpha, E_{-\alpha}] &= \frac{2\alpha \cdot H}{\alpha^2}, \quad \text{for any } \alpha \in \Delta, \\ [H, E_\alpha] &= \alpha E_\alpha. \end{aligned} \tag{2}$$

where the numbers $N_{\alpha,\beta}$ are nonzero for $\alpha + \beta \in \Delta$, and can all be taken as integers in the Chevalley basis.

It was indicated in refs. [4] and [9] that the Kac-Moody currents in the Feigin-Fuchs construction are closely related to the representation of a Lie algebra $\mathbf{g}$ on the space of sections R_Λ of a line bundle over a Schubert cell Y in the flag manifold $B_+ \backslash G$, determined by a character $\chi_\Lambda : B_+ \to \mathbf{C}$, where B_+ is a Borel subgroup generated by the positive roots and the Cartan subalgebra. A section is represented by the function f on G, which satisfies the following condition:

$$f(bg) = \chi_\Lambda(b) f(g), \quad \text{for} \quad b \in B_+ \text{ and } g \in G. \tag{3}$$

The character χ_Λ is defined as follows:

$$\chi_\Lambda(b) = \chi_\Lambda(\delta) = \mathrm{e}^{\phi\cdot\Lambda(H)}, \quad \text{for} \quad b = \zeta\delta,\ \delta = \mathrm{e}^{\phi\cdot H}. \tag{4}$$

Therefore $f(g)$ is parametrized by the coordinates z of Y and denoted as $f(z)$. A right representation σ_Λ of $\mathbf{g}$ on R_Λ is defined as ([12])

$$(\sigma_\Lambda(x)f)(z) = \Lambda((\mathrm{Ad}z)x)f(z) + \frac{d}{dt}f(z\mathrm{e}^{tx})\Big|_{t=0}, \tag{5}$$

where

$$\Lambda(y) = \begin{cases} 0, & \text{for } y \in \mathbf{n}_-, \\ \frac{d}{dt}\chi_\Lambda(\mathrm{e}^{ty})\Big|_{t=0}, & \text{for } y \in \mathbf{b}_+ = \mathbf{n}_+ \oplus \mathbf{h}. \end{cases} \tag{6}$$

In this representation $\sigma_\Lambda(x)$ becomes a differential operator with respect to z. Also we can define the representation of $\mathbf{n}_-$ from the left action of G on Y as follows. If we set, for $x \in \mathbf{n}_-$,

$$(\rho(x)f)(z) = \frac{d}{dt}f(e^{-tx}z)\Big|_{t=0}, \tag{7}$$

then ρ defines a representation of $\mathbf{n}_-$. As it was mentioned in ref. [9], ρ is related to the screening current $S(z)$, which satisfies

$$J(z)S(w) = \frac{\partial}{\partial w}\left(\frac{1}{z-w}O(w)\right), \tag{8}$$

where $J(z)$ is any Kac-Moody current and $O(z)$ is a local operator.

In the case of A_n it is useful to parametrize the coordinates of Y by the elements of the upper triangular matrix z in the Gauss decomposition (1). However for other simple Lie algebras this parametrization is awkward because all the elements of the upper-triangular matrix z in (1) are not independent. Moreover in the case of exceptional groups, we have to calculate group elements without appealing to the explicit matrix representation. Therefore in this letter we choose the following canonical parametrization of Y:

$$z = \exp(\sum_{\alpha\in\Delta_+} z_\alpha E_{-\alpha}). \tag{9}$$

In order to obtain the explicit expressions for currents, we need to calculate the product $z\mathrm{e}^{ty}$ ($y \in \mathbf{g}$). By using Hausdorff's formula $\exp(x)\exp(ty) = \exp(x + tu + O(t^2))$, where u is equal to

$$\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\left\{\sum_{p_1>0,\dots,p_m\geq 0}\frac{1}{p_1!\cdots p_m!}\frac{1}{p_1+\cdots+p_m+1}(\mathrm{ad}x)^{p_1+\cdots+p_m}y\right.$$

$$+\sum_{p_1>0,\ldots,p_{m-1}\geq 0}\frac{1}{p_1!\cdots p_{m-1}!}\frac{1}{p_1+\cdots+p_{m-1}+2}(\mathrm{ad}x)^{p_1+\cdots+p_{m-1}}(\mathrm{ad}y)x\Bigg\}, \quad (10)$$

or simply can be written in the following form:

$$u=\left(\frac{\mathrm{ad}x}{\mathrm{e}^{\mathrm{ad}x}-1}+\mathrm{ad}x\right)y=\left(\sum_{n=0}^{\infty}\frac{\tilde{B}_n}{n!}(\mathrm{ad}x)^n+\mathrm{ad}x\right)y, \quad (11)$$

The coefficients $\tilde{B}_n$ are defined as

$$\begin{aligned}\tilde{B}_0 &= 1,\quad \tilde{B}_1=-\frac{1}{2},\\ \tilde{B}_{2n} &= (-1)^{n-1}B_{2n},\quad \tilde{B}_{2n+1}=0, \text{for } n\geq 1,\end{aligned} \quad (12)$$

where B_n are Bernoulli numbers

$$B_2=\frac{1}{6}, B_4=\frac{1}{30},\ldots; B_{2n+1}=0 \quad \text{for } n\geq 1. \quad (13)$$

Substituting $x=\sum_{\beta\in\Delta_+}z_\beta E_{-\beta}$ and $y=E_{-\alpha}$ in (11), we get the representation for negative roots

$$\begin{aligned}\sigma_\Lambda(E_{-\alpha}) &= \sum_{\beta\in\Delta_+}N_{-\beta,-\alpha}z_\beta\frac{\partial}{\partial z_{\beta+\alpha}}\\ &+ \sum_{n=0}^{\infty}\frac{\tilde{B}_n}{n!}\sum_{\beta_1,\ldots,\beta_n\in\Delta_+}N_{-\beta_1,-\beta_2-\cdots-\alpha}\cdots N_{-\beta_n,-\alpha}z_{\beta_1}\cdots z_{\beta_n}\frac{\partial}{\partial z_{\beta_1+\cdots+\beta_n+\alpha}}.\end{aligned} \quad (14)$$

In a similar manner the representation ρ is obtained as follows:

$$\begin{aligned}\rho(E_{-\alpha}) &= \sum_{\beta\in\Delta_+}N_{-\beta,-\alpha}z_\beta\frac{\partial}{\partial z_{\beta+\alpha}}\\ &+ \sum_{n=0}^{\infty}(-1)^{n+1}\frac{\tilde{B}_n}{n!}\sum_{\beta_1,\ldots,\beta_n\in\Delta_+}N_{-\beta_1,-\beta_2-\cdots-\alpha}\cdots N_{-\beta_n,-\alpha}z_{\beta_1}\cdots z_{\beta_n}\frac{\partial}{\partial z_{\beta_1+\cdots+\beta_n+\alpha}}.\end{aligned} \quad (15)$$

If we substitute H for y in (11), we get the following Gauss decomposition:

$$\mathrm{e}^x\mathrm{e}^{tH}=\mathrm{e}^{tH}\mathrm{e}^{x+t(\mathrm{ad}x)H+O(t^2)}. \quad (16)$$

Therefore we get the currents for the Cartan part

$$\sigma_\Lambda(H) = \Lambda(H) + \sum_{\beta\in\Delta_+} \beta z_\beta \frac{\partial}{\partial z_\beta}. \tag{17}$$

Finally we consider the currents for the positive roots. In the case that α is a simple root, we obtain the Gauss decomposition:

$$\mathrm{e}^x \mathrm{e}^{tE_\alpha} = \mathrm{e}^{tE_\alpha} \mathrm{e}^{-tz_\alpha 2\alpha\cdot H/\alpha^2} \mathrm{e}^{x+tv+O(t^2)}, \tag{18}$$

where

$$v = \left(\frac{-\mathrm{ad}x}{\mathrm{e}^{-\mathrm{ad}x} - 1} - \mathrm{ad}x - 1 \right) z_\alpha \frac{2\alpha\cdot H}{\alpha^2} + \sum_{\beta,\beta-\alpha\in\Delta_+} z_\beta N_{-\beta,\alpha} E_{-\beta+\alpha}. \tag{19}$$

Therefore we get the following representation for the positive simple roots:

$$\begin{aligned} \sigma_\Lambda(E_\alpha) &= -\frac{2\Lambda\cdot\alpha}{\alpha^2} z_\alpha + \sum_{\beta,\beta-\alpha\in\Delta_+} N_{-\beta,\alpha} z_\beta \frac{\partial}{\partial z_{\beta-\alpha}} - \frac{1}{2} \sum_{\beta\in\Delta_+} \frac{2\alpha\cdot\beta}{\alpha^2} z_\alpha z_\beta \frac{\partial}{\partial z_\beta} \\ &+ \sum_{n=2}^{\infty} \frac{\tilde{B}_n}{n!} \sum_{\beta_1,\dots,\beta_n\in\Delta_+} \frac{2\alpha\cdot\beta_n}{\alpha^2} N_{-\beta_1,-\beta_2-\cdots-\beta_n} \cdots N_{-\beta_{n-1},-\beta_n} z_\alpha z_{\beta_1} \cdots z_{\beta_n} \frac{\partial}{\partial z_{\beta_1+\cdots+\beta_n}}. \end{aligned} \tag{20}$$

For the positive roots α except for the simple roots we must use more complicated Gauss decomposition formulas instead of (18). However the representations corresponding to the positive non-simple roots can be obtained by using the commutation relations of the generators for the simple roots.

Thus we obtain the representation of simple Lie algebras $\sigma_\Lambda(E_{-\alpha})$, $\sigma_\Lambda(E_\alpha)$, $\sigma_\Lambda(H^i)$ and $\rho(E_{-\alpha})$. The prescription † to extend this to the affine case is as follows ([9]):

1. Replace $\partial/\partial z_\alpha \to \beta_\alpha(z)$ and $z_\alpha \to \gamma_\alpha(z)$ ($\alpha \in \Delta_+$), where β_α and γ_α are the bosonic ghosts with conformal dimensions 1 and 0 respectively, and satisfy

$$\beta_\alpha(z)\gamma_{\alpha'}(w) = \frac{\delta_{\alpha,\alpha'}}{z-w}, \quad \alpha,\alpha'\in\Delta_+. \tag{21}$$

† In fact this prescription can be proved by the extension of the present finite-dimensional flag manifold analysis to the infinite-dimensional one as discussed by Bernard and Felder for the $A_1^{(1)}$ case([7]).

2. Replace Λ^i by the term $-\mathrm{i}\alpha_+\partial\varphi^i(z)$ for the Cartan part and the positive roots. Add the term $a_\alpha\partial\gamma_\alpha(z)$ (a_α is a number determined by the consistency of the operator product relations of the affine Lie algebra $\hat{\mathbf{g}}$) to the currents corresponding to the simple roots α. Multiply $\rho(E_{-\alpha})$ by $\exp(\mathrm{i}\alpha_-\alpha\varphi(z))$ for the screening current. Here $\varphi^i(z)$ $(i=1,\ldots,r)$ are free bosons satisfying

$$\varphi^i(z)\varphi^j(w) = -\delta^{ij}\log(z-w), \tag{22}$$

and $\alpha_+ \equiv \sqrt{k+g}$ and $\alpha_- = -1/\alpha_+$, where k is the level of the affine Lie algebra and g is the dual Coxeter number of the algebra.

By this prescription we get the general expressions for the Kac-Moody currents of the affine Lie algebras which include not only A,B,C,D types but also the exceptional types. For the negative root, the corresponding Kac-Moody currents are

$$\begin{aligned} J_{-\alpha}(z) &= \beta_\alpha + \frac{1}{2}\sum_{\beta_1\in\Delta_+} N_{-\beta_1,-\alpha}\gamma_{\beta_1}\beta_{\beta_1+\alpha} \\ &+ \sum_{n=2}^{\infty}\frac{\tilde{B}_n}{n!}\sum_{\beta_1,\ldots,\beta_n\in\Delta_+} N_{-\beta_1,-\beta_2-\cdots-\alpha}\cdots N_{-\beta_n,-\alpha}\gamma_{\beta_1}\cdots\gamma_{\beta_n}\beta_{\beta_1+\cdots+\beta_n+\alpha}. \end{aligned} \tag{23}$$

For the positive simple roots, the currents are

$$\begin{aligned} J_\alpha(z) &= a_\alpha\partial\gamma_\alpha + \frac{2\mathrm{i}\alpha_+\gamma_\alpha\alpha\cdot\partial\varphi}{\alpha^2} - \frac{1}{2}\sum_{\beta_1\in\Delta_+}\frac{2\alpha\cdot\beta_1}{\alpha^2}\gamma_\alpha\gamma_{\beta_1}\beta_{\beta_1} + \sum_{\substack{\beta\in\Delta_+ \\ \beta-\alpha\in\Delta_+}} N_{-\beta,\alpha}\gamma_\beta\beta_{\beta-\alpha} \\ &+ \sum_{n=2}^{\infty}\frac{\tilde{B}_n}{n!}\sum_{\beta_1,\ldots,\beta_n\in\Delta_+}\frac{2\alpha\cdot\beta_n}{\alpha^2}N_{-\beta_1,-\beta_2-\cdots-\beta_n}\cdots N_{-\beta_{n-1},-\beta_n}\gamma_\alpha\gamma_{\beta_1}\cdots\gamma_{\beta_n}\beta_{\beta_1+\cdots+\beta_n}. \end{aligned} \tag{24}$$

The constant a_α is determined by the operator product expansion between J_α and $J_{-\alpha}$

$$J_\alpha(z)J_{-\alpha}(w) = \frac{2k/\alpha^2}{(z-w)^2} + \frac{2\alpha\cdot H(w)/\alpha^2}{z-w} + \cdots, \tag{25}$$

and is given as

$$a_\alpha = \frac{2k}{\alpha^2} + \frac{1}{2}\sum_{\beta_1\in\Delta_+,\beta_1-\alpha\in\Delta_+} N_{-\beta_1,\alpha}N_{-\beta_1+\alpha,-\alpha}. \tag{26}$$

We note that the first three terms in eq. (24) are the same form of those of the $A_1^{(1)}$ Kac-moody current for the positive root. However it should be noted that the above Kac-Moody currents (23), (24) are different from those appeared in other papers ([4],[5],[6],[7], [8], [9],[10]) because of the parametrization (9). The currents for the Cartan part are

$$H^i(z) = -\mathrm{i}\alpha_+\partial\varphi^i(z) + \sum_{\alpha\in\Delta_+}\alpha^i\gamma_\alpha\beta_\alpha(z), \quad (i=1,\ldots,r). \tag{27}$$

The screening operator, which corresponds to the simple root α, is

$$\begin{aligned} S_\alpha(z) &= (\beta_\alpha - \frac{1}{2}\sum_{\beta\in\Delta_+} N_{-\beta,-\alpha}\gamma_\beta\beta_{\beta+\alpha} \\ &+ \sum_{n=2}^{\infty}\frac{\tilde{B}_n}{n!}\sum_{\beta_1,\ldots,\beta_n\in\Delta_+} N_{-\beta_1,-\beta_2-\cdots-\alpha}\cdots N_{-\beta_n,-\alpha}\gamma_{\beta_1}\cdots\gamma_{\beta_n}\beta_{\beta_1+\cdots+\beta_n+\alpha})\mathrm{e}^{\mathrm{i}\alpha_-\alpha\cdot\varphi}. \end{aligned} \tag{28}$$

The above Kac-Moody currents should satisfy the operator product expansions of the affine Lie algebra

$$\begin{aligned} J_\alpha(z)J_\beta(w) &= \frac{N_{\alpha,\beta}J_{\alpha+\beta}(w)}{z-w} + \cdots, \qquad \text{for } \alpha+\beta\in\Delta, \\ J_\alpha(z)J_{-\alpha}(w) &= \frac{2k/\alpha^2}{(z-w)^2} + \frac{2\alpha\cdot H(w)/\alpha^2}{z-w} + \cdots, \\ H^i(z)J_\alpha(w) &= \frac{\alpha^i J_\alpha(w)}{z-w} + \cdots, \\ H^i(z)H^j(w) &= \frac{k\delta_{ij}}{(z-w)^2} + \cdots. \end{aligned} \tag{29}$$

In order to check (29) we need some identities among the structure constants $N_{\alpha,\beta}$. It is difficult to prove these operator product expansions in general. However (29) can be checked explicitly for some nontrivial cases as discussed below. Note that the currents for the positive non-simple roots can be obtained from the above operator product expansions. They are essentially similar to (24) but we need the terms which come from the operator product expansions between $a_\alpha\partial\gamma_\alpha$ and the other terms in (24).

We will show some examples for the Feigin-Fuchs representations of the affine Lie algebras. Detailed analysis for the general affine Lie algebras will be discussed in a subsequent paper.

(1) The $B_2^{(1)}$-type affine Lie algebra ($\dim B_2 = 10,\ g = 3$). We denote the simple roots of B_2 by α_1 and α_2 (α_1 is the long root). Currents for negative (simple) roots are

$$\begin{aligned} J_{-\alpha_1}(z) &= \beta_1 - \frac{1}{2}\gamma_2\beta_{12} + \frac{1}{6}\gamma_2^2\beta_{122}, \\ J_{-\alpha_2}(z) &= \beta_2 + \frac{1}{2}\gamma_1\beta_{12} + \Big(\gamma_{12} - \frac{1}{6}\gamma_1\gamma_2\Big)\beta_{122}, \end{aligned} \tag{30}$$

where we define β_1, β_{12} as β_{α_1}, $\beta_{\alpha_1+\alpha_2}$ etc. Currents for positive simple roots are

$$\begin{aligned} J_{\alpha_1}(z) &= \Big(k + \frac{1}{2}\Big)\partial\gamma_1 + \mathrm{i}\alpha_+\gamma_1\alpha_1\cdot\partial\varphi - \gamma_1^2\beta_1 \\ &+ \Big(-\gamma_{12} + \frac{1}{2}\gamma_1\gamma_2\Big)\beta_2 + \Big(-\frac{1}{2}\gamma_{12} - \frac{1}{4}\gamma_1\gamma_2\Big)\gamma_1\beta_{12} - \frac{1}{3}\gamma_1\gamma_2\gamma_{12}\beta_{122}, \\ J_{\alpha_2}(z) &= (2k+2)\partial\gamma_2 + 2\mathrm{i}\alpha_+\gamma_2\alpha_2\cdot\partial\varphi - \gamma_2^2\beta_2 \\ &+ \Big(2\gamma_{12} + \gamma_1\gamma_2\Big)\beta_1 + \Big(\gamma_{122} + \frac{1}{3}\gamma_1\gamma_2^2\Big)\beta_{12} + \Big(-\gamma_{122} + \frac{1}{3}\gamma_2\gamma_{12}\Big)\gamma_2\beta_{122}. \end{aligned} \tag{31}$$

Screening currents are

$$\begin{aligned} S_{\alpha_1}(z) &= \Big(\beta_1 + \frac{1}{2}\gamma_2\beta_{12} + \frac{1}{6}\gamma_2^2\beta_{122}\Big)\mathrm{e}^{\mathrm{i}\alpha_-\alpha_1\cdot\varphi}, \\ S_{\alpha_2}(z) &= \Big[\beta_2 - \frac{1}{2}\gamma_1\beta_{12} + \Big(-\gamma_{12} - \frac{1}{6}\gamma_1\gamma_2\Big)\beta_{122}\Big]\mathrm{e}^{\mathrm{i}\alpha_-\alpha_2\cdot\varphi}. \end{aligned} \tag{32}$$

(2) The $G_2^{(1)}$-type affine Lie algebra ($\dim G_2 = 14,\ g = 4$). We denote the simple roots by α_1 and α_2 (α_1 is the long root). Currents for negative (simple) roots are

$$\begin{aligned} J_{-\alpha_1}(z) &= \beta_1 - \frac{1}{2}\gamma_2\beta_{12} + \frac{1}{6}\gamma_2^2\beta_{112} + \Big(\frac{1}{2}\gamma_{1222} + \frac{1}{4}\gamma_2\gamma_{122} - \frac{1}{120}\gamma_1\gamma_2^3\Big)\beta_{11222}, \\ J_{-\alpha_2}(z) &= \beta_2 + \frac{1}{2}\gamma_1\beta_{12} + \Big(\gamma_{12} - \frac{1}{6}\gamma_1\gamma_2\Big)\beta_{122} + \Big(\frac{3}{2}\gamma_{122} - \frac{1}{2}\gamma_2\gamma_{12}\Big)\beta_{1222} \\ &+ \Big(-\frac{1}{2}\gamma_1\gamma_{122} + \frac{1}{2}\gamma_{12}^2 + \frac{1}{120}\gamma_1^2\gamma_2^2\Big)\beta_{11222}. \end{aligned} \tag{33}$$

Currents for positive simple roots are

$$\begin{aligned} J_{\alpha_1}(z) &= (k+1)\partial\gamma_1 + \mathrm{i}\alpha_+\gamma_1\alpha_1\cdot\partial\varphi - \gamma_1^2\beta_1 \\ &+ \Big(-\gamma_{12} + \frac{1}{2}\gamma_1\gamma_2\Big)\beta_2 + \Big(-\frac{1}{2}\gamma_{12} - \frac{1}{4}\gamma_1\gamma_2\Big)\gamma_1\beta_{12} - \frac{1}{3}\gamma_1\gamma_2\gamma_{12}\beta_{122} \\ &+ \Big(\gamma_{11222} + \frac{1}{2}\gamma_1\gamma_{1222} - \frac{1}{4}\gamma_1\gamma_2\gamma_{122} + \frac{1}{40}\gamma_1^2\gamma_2^3\Big)\beta_{1222} \end{aligned}$$

$$
\begin{aligned}
&+ \quad \Big(-\frac{1}{2}\gamma_{11222}+\frac{1}{4}\gamma_1\gamma_{1222}-\frac{1}{4}\gamma_{12}\gamma_{122}-\frac{1}{120}\gamma_1\gamma_2^2\gamma_{12}\Big)\gamma_1\beta_{11222},\\
J_{\alpha_2}(z) \quad &= \quad (3k+5)\partial\gamma_2+3\mathrm{i}\alpha_+\gamma_2\alpha_2\cdot\partial\varphi-\gamma_2^2\beta_2\\
&+ \quad \Big(3\gamma_{12}+\frac{3}{2}\gamma_1\gamma_2\Big)\beta_1+\Big(2\gamma_{122}+\frac{1}{2}\gamma_2\gamma_{12}+\frac{5}{12}\gamma_1\gamma_2^2\Big)\beta_{12}\\
&+ \quad \Big(\gamma_{1222}-\frac{1}{2}\gamma_2\gamma_{122}+\frac{1}{2}\gamma_2^2\gamma_{12}\Big)\beta_{122}+\Big(-\frac{3}{2}\gamma_{1222}+\frac{1}{4}\gamma_2\gamma_{122}-\frac{1}{24}\gamma_1\gamma_2^3\Big)\gamma_2\beta_{1222}\\
&+ \quad \Big(-\frac{1}{2}\gamma_1\gamma_{1222}+\frac{1}{2}\gamma_{12}\gamma_{122}+\frac{1}{60}\gamma_1\gamma_2^2\gamma_{12}\Big)\gamma_2\beta_{11222}. \qquad (34)
\end{aligned}
$$

The currents for non-simple roots can be obtained by making operator products of these two currents. Screening currents are

$$
\begin{aligned}
S_{\alpha_1}(z) \quad &= \quad \Big[\beta_1+\frac{1}{2}\gamma_2\beta_{12}+\frac{1}{6}\gamma_2^2\beta_{122}+\Big(-\frac{1}{2}\gamma_{1222}+\frac{1}{4}\gamma_2\gamma_{122}-\frac{1}{120}\gamma_1\gamma_2^3\Big)\beta_{11222}\Big]\mathrm{e}^{\mathrm{i}\alpha_-\alpha_1\cdot\varphi},\\
S_{\alpha_2}(z) \quad &= \quad \Big[\beta_2-\frac{1}{2}\gamma_1\beta_{12}+\Big(-\gamma_{12}-\frac{1}{6}\gamma_1\gamma_2\Big)\beta_{122}+\Big(-\frac{3}{2}\gamma_{122}-\frac{1}{2}\gamma_2\gamma_{12}\Big)\beta_{1222}\\
&+ \quad \Big(-\frac{1}{2}\gamma_1\gamma_{122}+\frac{1}{2}\gamma_{12}^2+\frac{1}{120}\gamma_1^3\gamma_2^2\Big)\beta_{11222}\Big]\mathrm{e}^{\mathrm{i}\alpha_-\alpha_2\cdot\varphi}. \qquad (35)
\end{aligned}
$$

Though these expressions are quite cumbersome to manipulate, one can explicitly check that they completely satisfy the operator product relations of the affine Lie algebras. One can also see that the outer-automorphism symmetry of the Lie algebra in the currents is manifest in our expression. In particular the $\mathbf{Z}_3$ symmetry appears in the case of $D_4^{(1)}$. By using the screening charges we can study the null field structure of the Fock modules of the affine Lie algebras. The Feigin-Fuchs construction enables us to calculate correlation functions on the Riemann surfaces as integral representations. It may be interesting to study the monodromy properties of correlation functions using this representation and to investigate the relationship between the WZW models and the quantum groups. In particular the above expressions for the affine Lie algebra $G_2^{(1)}$ (33), (34), (35) are the first results obtained for the Feigin-Fuchs representations of the exceptional affine Lie algebra. One can use these expressions to calculate the braid matrices and to study the correspondence with the quantum R matrix for G_2, which has already been obtained ([13]). It is also important to study the coset construction ([10]) and the related W-algebras. The W-algebras corresponding to the affine Lie algebras $A_n^{(1)}$, $B_n^{(1)}$ and $D_n^{(1)}$ are known ([14]). It is expected that the W-algebra for the affine Lie algebra $G_2^{(1)}$ has a spin-6 current together with the energy-momentum tensor, and that these two currents form a closed algebra. We also get the expressions for the generalized parafermionic currents by the bosonization of the bosonic ghosts (β, γ) and factorizing the Cartan part ([8]). To analyze the parafermionic Fock modules, we need the fermionic screening operators. We emphasize that the geometrical method is a powerful tool for the free field construction of the affine Lie algebras. It seems interesting to study the coset models and their chiral algebras from this geometrical point of view. These problems will be discussed elsewhere.

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References

[1] M. Wakimoto, Commun. Math. Phys. **92** (1984) 455.

[2] A. B. Zamolodchikov, unpublished.

[3] M. Bershadsky and H. Ooguri, Commun. Math. Phys. **126** (1989) 429.

[4] B. L. Feigin and E. V. Frenkel, Russ. Math. Surv. **43** (1989) 221; in "Physics and Mathematics of Strings" (eds. L. Brink et al., World Scientific, 1990) 271; Commun. Math. Phys. **128** (1990) 161, Lett. Math. Phys. **19** (1990) 307.

[5] A. Gerasimov, A. Marshakov, A. Morozov, M. Olshanetsky and S. Shatashvili, Moscow ITEP preprints 64-89, 70-89, 72-89, 74-89, March 1989.

[6] D. Nemeschansky, Phys. Lett. **224B** (1989) 121.
A. Bilal, Phys. Lett. **226B** (1989) 272.
J. Distler and Z. Qiu, Nucl. Phys. **B336** (1990) 533.
K. Ito, Phys. Lett. **230B** (1989) 71; Nucl. Phys. **B332** (1990) 566.

[7] D.Bernard and G.Felder, Commun. Math. Phys. **127** (1990) 145.

[8] K. Ito and Y. Kazama, Mod. Phys. Lett. **A5** (1990) 215.

[9] P. Bouwknegt, J. McCarthy and K. Pilch, MIT preprint CTP#1797, October 1989.

[10] M. Kuwahara and H. Suzuki, Phys. Lett. **235B** (1990) 52.

[11] M. Kuwahara, N. Ohta and H. Suzuki, Osaka preprint OS-GE 05-89, October 1989.

[12] B. Kostant, Lect. Notes Math. **466** (1974) 101.

[13] A. Kuniba, J. Phys. **A 23** (1990) 1349.

[14] S. L. Lukyanov and V. A. Fateev, Kiev preprints ITP-88-74P, 75P, 76P, 1988.