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Quantum Superalgebra $U_q osp(2,2)$

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ABSTRACT

We construct the quantum universal enveloping algebra $U_q osp(2,2)$ and calculate quantum universal R matrix. We obtain a new solvable model associated with the algebra osp(2,2).

Recent studies on the connection between conformal field theories and exactly solvable models present various suggestive problems in physics and mathematics. Many similar structures appear in both theories and phenomenologically they are closely related to quantum group or the q-deformation of the universal enveloping algebra [1]. The quantum groups associated with non-exceptional types of Lie algebras [2,3] were studied by many authors. Construction of the quantum groups associated with the super Lie algebras [4] is an important step to study connections among the algebras, exactly solvable models, conformal field theories and string theories. From the string theoretical point of view, it is important to find the exactly solvable models corresponding to N=2 superconformal field theories which represent the classical vacuum in the compactifications of string theories.In the case of $U_q osp(1,2)$ and related solvable models corresponding to $N=1$ minimal models are discussed in refs. [5,6]. For the super Lie algebra of the type $A(m,n)$, the quantum supergroup [7] and solvable models [8] have also been discussed.

In this note we construct the q-deformation of the universal enveloping algebra $U_q osp(2,2)$ of the superalgebra $osp(2,2)$ and study its representation. We also construct a supersymmetric solvable model in two dimensions associated with $U_q osp(2,2)$.

The superalgebra $osp(2,2)$ is a subalgebra of $N=2$ superconformal algebra of NS sector which is defined as follows:

$$\begin{aligned}
[L_m, L_n] =&(m-n)L_{m+n} + \frac{c}{4}m(m^2-1)\delta_{m+n,0},\\
[L_m, T_n] =& -nT_{m+n},\\
[L_m, G_r] =&(\frac{1}{2}m-r)G_{m+r}, \qquad\qquad [L_m, \bar{G}_r] = (\frac{1}{2}m-r)\bar{G}_{m+r},\\
[T_m, T_n] =&cm\delta_{m+n,0},\\
[T_m, G_r] =&G_{m+r}, \qquad\qquad [T_m, \bar{G}_n] = -\bar{G}_{m+r},\\
\{G_r, \bar{G}_s\} =&L_{r+s} + \frac{1}{2}(r-s)T_{r+s} + \frac{1}{2}c(r^2-\frac{1}{4})\delta_{r+s,0},\\
\{G_r, G_s\} =&\{\bar{G}_r, \bar{G}_s\} = 0.
\end{aligned} \tag{1}$$

The elements $L_0, L_{\pm 1}$, T_0, $G_{\pm\frac{1}{2}}$ and $\bar{G}_{\pm\frac{1}{2}}$ form the subalgebra $osp(2,2)$. Let us define the generators H, $J_\pm$, T, $V_\pm$ and $\bar{V}_\pm$ as $H = -L_0$, $J_\pm = \pm L_{\pm 1}$, $T = -T_0$, $V_\pm = \frac{1}{\sqrt{2}} G_{\frac{\pm 1}{2}}$, $\bar{V}_\pm = \frac{1}{\sqrt{2}} \bar{G}_{\pm\frac{1}{2}}$. From (1)and the above definitions, the $osp(2,2)$ superalgebra is expressed as follows:

$$[H, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = 2H,$$

$$\begin{aligned}
[H \pm \frac{1}{2}T, V_\pm] &= 0, \qquad [H \mp \frac{1}{2}T, V_\pm] = \pm V_\pm, \\
[H \pm \frac{1}{2}T, \bar{V}_\pm] &= \pm \bar{V}_\pm, \qquad [H \mp \frac{1}{2}T, \bar{V}_\pm] = 0,
\end{aligned}$$

$$\{V_i, V_j\} = \{\bar{V}_i, \bar{V}_j\} = 0 \quad (i, j = \pm),$$

$$\{V_+, \bar{V}_-\} = -\frac{1}{2}(H + \frac{1}{2}T), \qquad \{\bar{V}_+, V_-\} = -\frac{1}{2}(H - \frac{1}{2}T),$$

$$\{V_\pm, \bar{V}_\pm\} = \pm \frac{1}{2} J_\pm,$$

Let us consider the finite dimensional representation of $osp(2,2)$. The irreducible representations are parametrized by the $sl(2, \mathbf{C})$ spin of an integer or half-integer $j \in \mathbf{N}/2$. Firstly we discuss the fundamental representation which corresponds to spin $j = \frac{1}{2}$. It is shown as follows;

$$\begin{aligned}
V_+ &= \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \bar{V}_- = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
\bar{V}_+ &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad V_- = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\
H &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{2}$$

In (2), the basis states are $\left|\frac{1}{2},\frac{1}{2}\right\rangle$, $\left|\frac{1}{2},-\frac{1}{2}\right\rangle$, $\left|-\frac{1}{2},\frac{1}{2}\right\rangle$, and $\left|-\frac{1}{2},-\frac{1}{2}\right\rangle$, where the state $\left|m,m'\right\rangle$ is defined as

$$(H+\frac{T}{2})\left|m,m'\right\rangle = m\left|m,m'\right\rangle, \qquad (H-\frac{T}{2})\left|m,m'\right\rangle = m'\left|m,m'\right\rangle. \tag{3}$$

We call the states $\left|\frac{1}{2},\frac{1}{2}\right\rangle$ and $\left|-\frac{1}{2},-\frac{1}{2}\right\rangle$ even (bosonic), and $\left|\frac{1}{2},-\frac{1}{2}\right\rangle$ and $\left|-\frac{1}{2},\frac{1}{2}\right\rangle$ odd(fermionic).

The universal enveloping algebra $Uosp(2,2)$ is generated by $H,T,V_\pm,\bar{V}_\pm$. We define the quantum universal enveloping algebra $U_qosp(2,2)$ by replacing the anti-commutation relations for $V_\pm,\bar{V}_\pm$ as follows;

$$\{V_+,\bar{V}_-\} = -\frac{1}{4}[2P_+]_q, \qquad \{\bar{V}_+,V_-\} = -\frac{1}{4}[2P_-]_q, \tag{4}$$

where $P_\pm = H \pm \frac{1}{2}T$, and $[x]_q$ is the usual q-analog of x;

$$[x]_q = \frac{q^x - q^{-x}}{q-q^{-1}}. \tag{5}$$

Note ,for instance,$[P_\pm, V_\pm] = 0$,or $[P_\mp, V_\pm] = \pm V_\pm$ as usual, and the $q-$analogue of $J_\pm$ are decided by the analogues of $H,T,V,\bar{V}$. The q-deformed Casimir element is

$$\begin{aligned} C = &-4\{\bar{V}_-,V_-\}\{V_+,\bar{V}_+\} + ([2P_+]_q - [2P_+ + 2]_q)V_-\bar{V}_+ \\ &+ ([2P_-]_q - [2P_- + 2]_q)\bar{V}_-V_+ + \frac{1}{4}[2P_+]_q[2P_-]_q, \end{aligned} \tag{6}$$

which acts on a highest weight state $\left|j\right\rangle$ as

$$C\left|j\right\rangle = \frac{1}{4}[2j]^2\left|j\right\rangle. \tag{7}$$

We introduce the coproduct Δ which is a homomorphism from $U_qosp(2,2)$ to $U_qosp(2,2)\otimes U_qosp(2,2)$, $\Delta : U_q \to U_q \otimes U_q$, by

$$\begin{aligned} \Delta(V_\pm) =& q^{P_\pm}\otimes V_\pm + V_\pm \otimes q^{-P_\pm}, \\ \Delta(\bar{V}_\pm) =& q^{P_\mp}\otimes \bar{V}_\pm + \bar{V}_\pm \otimes q^{-P_\mp}, \\ \Delta(P_\pm) =& P_\pm \otimes 1 + 1 \otimes P_\pm, \end{aligned} \tag{8}$$

It is straightforward to check that (8)is compatible with the algebra of $U_qosp(2,2)$.

Another coproduct $\bar{\Delta}$ can be defined as $\bar{\Delta} = \sigma \circ \Delta$, where σ is a permutation on $U_q \otimes U_q$, which is given by replacing q to q^{-1}. The relation between Δ and $\bar{\Delta}$ is given by the universal R matrix which act on $U_q \otimes U_q$, as follows:

$$\bar{\Delta} R = R\Delta. \tag{9}$$

From (4)and (8)we find

$$\begin{aligned} R =& q^{2(P_+\otimes P_- + P_-\otimes P_+)}((\sum_{n=1}^{\infty} c_n q^{-n(P_+ + P_-)} \otimes q^{n(P_+ + P_-)}(q^{2n}(Y_-^n \otimes X_+^n + X_-^n \otimes Y_+^n) \\ &+ (X_-^n \otimes X_+^n + Y_-^n \otimes Y_+^n) \\ &- 4q^{-n}(q-q^{-1}) \\ &\times ((q^{-P_+} \otimes q^{P_+})(\bar{V}_- Y_-^n \otimes V_+ X_+^n) + (q^{-P_-} \otimes q^{P_-})(V_- X_-^n \otimes \bar{V}_+ Y_+^n))) \\ &- 4(q-q^{-1})((q^{-P_+} \otimes q^{P_+})(\bar{V}_- \otimes V_+) + (q^{-P_-} \otimes q^{P_-})(V_- \otimes \bar{V}_+)) + 1). \end{aligned} \tag{10}$$

where $X_\pm = \bar{V}_\pm V_\pm$, $Y_\pm = V_\pm \bar{V}_\pm$ and

$$c_n = (-1)^n \prod_{i=1}^{n} \frac{16(q-q^{-1})^2}{q^{4i}-1}. \tag{11}$$

In our choice the matrix R is a lower triangular matrix.

Because $[\Delta(P_\pm), R] = 0$, its irreducible representations are labelled by total P_+ and P_-.In particular, for spin $j = \frac{1}{2}$ representation R takes the following form (For simplicity we denote $\left|\frac{1}{2},\frac{1}{2}\right\rangle, \left|-\frac{1}{2},-\frac{1}{2}\right\rangle, \left|\frac{1}{2},-\frac{1}{2}\right\rangle, \left|-\frac{1}{2},\frac{1}{2}\right\rangle$ by b_1, b_2, f_1, f_2, respectively.)

(I) 1-dimensional representations are $(P_+, P_-) = (+1,+1)$, $(-1,-1)$, $(+1,-1)$, $(-1,+1)$, which correspond to $b_1 \otimes b_1$,$b_2 \otimes b_2$,$f_1 \otimes f_1$, and $f_2 \otimes f_2$.

$$R = q \qquad \text{for} \;\; b_1 \otimes b_1 \;\; \text{and} \;\; b_2 \otimes b_2 \tag{12}$$

$$R = q^{-1} \qquad \text{for} \;\; f_1 \otimes f_1 \;\; \text{and} \;\; f_2 \otimes f_2 \tag{13}$$

(II) 2-dimensional representations are $(P_+, P_-) = (+1,0)$, $(0,+1)$, $(-1,0)$,

$(0,-1)$, which correspond to $(b_1\otimes f_1, f_1\otimes b_1)$, $(b_1\otimes f_2, f_2\otimes b_1)$, $(f_1\otimes b_2, b_2\otimes f_1)$, $(f_2\otimes b_2, b_2\otimes f_2)$;

$$R=\begin{pmatrix} 1 & 0 \\ q-q^{-1} & 1 \end{pmatrix}, \tag{14}$$

for $(b_1\otimes f_1, f_1\otimes b_1)$, $(b_1\otimes f_2, f_2\otimes b_1)$, $(f_1\otimes b_2, b_2\otimes f_1)$, $(f_2\otimes b_2, b_2\otimes f_2)$. (III) 4-dimensional representation is $P_+=P_-=0$, whose basis is $b_1\otimes b_2$,$f_1\otimes f_2$,$f_2\otimes f_1$,$b_2\otimes b_1$;

$$R=\begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ q-q^{-1} & q & 0 & 0 \\ q-q^{-1} & 0 & q & 0 \\ 2(q-q^{-1}) & q-q^{-1} & q-q^{-1} & q^{-1} \end{pmatrix} \tag{15}$$

for $(b_1\otimes b_2, f_1\otimes f_2, f_2\otimes f_1, b_2\otimes b_1)$.

The tensor product of two spin $\frac{1}{2}(C=\frac{1}{4})$ representations can be decomposed into the direct sum of the spin $j=1(C=\frac{1}{4}[2]^2)$ and the spin $j=0(C=0)$ representations. We denote the projector onto spin l as P_l. We also define the permutation π such that

$$\pi\left|\alpha\right\rangle\otimes\left|\beta\right\rangle=(-)^{p(\alpha)p(\beta)}\left|\beta\right\rangle\otimes\left|\alpha\right\rangle, \tag{16}$$

where $p(\alpha)=1(0)$ if $|\alpha\rangle$ is odd (even). We can show that

$$\check{R}^{\frac{1}{2},\frac{1}{2}}=-q^{-1}P_0+qP_1, \tag{17}$$

where $\check{R}=\pi\circ R$. $\check{R}$ is a solution of the braid equation;

$$\check{R}_{12}\check{R}_{23}\check{R}_{12}=\check{R}_{23}\check{R}_{12}\check{R}_{23}. \tag{18}$$

Let us introduce the spectral parameter dependent $\check{R}(x)$ matrix according to the method by Jimbo [3]. For the spin $\frac{1}{2}$ representation the R matrix $\check{R}(x)$ becomes

$$\check{R}^{\frac{1}{2},\frac{1}{2}}(x) = P_0 + \frac{x-q^2}{1-xq^2}P_1, \tag{19}$$

which satisfies the Yang - Baxter equation

$$\check{R}^{\frac{1}{2},\frac{1}{2}}(x)_{12}\check{R}^{\frac{1}{2},\frac{1}{2}}(xy)_{23}\check{R}^{\frac{1}{2},\frac{1}{2}}(y)_{12} = \check{R}^{\frac{1}{2},\frac{1}{2}}(y)_{23}\check{R}^{\frac{1}{2},\frac{1}{2}}(xy)_{12}\check{R}^{\frac{1}{2},\frac{1}{2}}(x)_{23} \tag{20}$$

and

$$\check{R}^{\frac{1}{2},\frac{1}{2}}(1) = 1, \check{R}^{\frac{1}{2},\frac{1}{2}}(0) = -q\check{R}^{\frac{1}{2},\frac{1}{2}}, \check{R}^{\frac{1}{2},\frac{1}{2}}(\infty) = -q^{-1}(\check{R}^{\frac{1}{2},\frac{1}{2}})^{-1}. \tag{21}$$

Therefore the two-dimensional model (vertex model) whose Boltzmann weights are given by $\check{R}(x)$ is solvable. We note that the operator $R(x) = \pi \circ \check{R}(x)$ satisfies the graded Yang-Baxter equation [9].

In terms of the projectors the spectral decomposition ((19)) has the same form with that for the case of $U_q sl(2, \mathbf{C})$. But the matrix elements (Boltzmann weights) calculated from (12) - (15) are different from those of $U_q sl(2, \mathbf{C})$ (the 6-vertex model).

The higher spin representations can be discussed in a similar manner. We note that the existence of $\check{R}(x)$ for higher spin representations can also be shown by using the method of fusion [10] (or Z-invariance, composition). The point is the following. The matrix $\check{R}$ has only two different eigenvalues.Therefore it satisfies the Hecke algebra and the projectors can be expressed in terms of the generators of the Hecke algebra [11].

In this paper we have presented a brief account for the case that the parameter q is generic. The representation when q is a root of unity, the construction of the associated lattice models and the relations to conformal field theories will be discussed in a subsequent paper.

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