ODE/IM correspondence and modified affine Toda field equations

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KI and C. Locke, arXiv:1312.6759, 1502.00906

Introduction

- The ODE/IM correspondence is a relation between spectral analysis of ordinary differential equation (ODEs), and the "functional relations" approach to 2d quantum integrable model (IM). [Dorey-Tateo]
- This is an example of the correspondence between classical and quantum integrable models
- has many applications
  - gluon scattering amplitudes/null polygonal Wilson loops in $\mathcal{N}=4$ SYM at strong coupling and minimal surface in AdS spacetime [Alday-Maldacena]
  - BPS spectrum in $N=2$ SUSY gauge theories [Gaiotto-Moore-Neitzke]
  - PT-symmetric quantum mechanics $\mathcal{H}=p^2+ix^3$
  - $t-t^*$ equations in SUSY field theories

- Dorey-Tateo (1998) studied the spectral determinant of the quartic potential and its relation to the A_3-related Y-system.
- ODE/IM correpsondence for classical Lie algebras Dorey-Dunning-Masoero-Suzuki-Tateo, 2006
- Lukyanov-Zamolodchikov (2010) studied the linear problem associated with the modified sinh-Gordon equation in the context of ODE/IM correspondence ($A_1^{(1)}$: $\varphi_{tt} - \varphi_{xx} + \sinh\varphi = 0$)
- The results were generalized to the case of Tzitzéica-Bullough -Dodd equation by Dorey et al. (2012).($A_2^{(2)}$: $\varphi_{tt} - \varphi_{xx} + e^{2\varphi} - e^{-\varphi} = 0$)

We will

- Introduce the affine Toda field equation and its linear problem
- Discuss the conformal limit and its relation to the ODE/IM correspondence
- Study the Bethe ansatz equations for affine Lie algebras

The general scheme of the ODE/IM correspondence for affine Toda equation is [Dorey-Faldella-Negro-Tateo]

ODE	$\Longleftrightarrow$ ODE/IM	BAE massless TBA	$\Longleftrightarrow$	CFT
$\Uparrow$ Conformal limit		$\Uparrow$ UV limit		
Linear problem	$\Longleftrightarrow$ massive ODE/IM	BAE massive TBA	$\Longleftrightarrow$	massive QFT
$\Uparrow$				
affine Toda equation				

OED/IM correspondence

[Dorey-Tateo, Bazhanov-Lukyanov-Zamolodchikov]

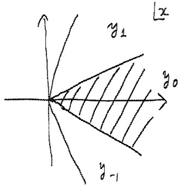
- ODE

$$\left[-\frac{d^2}{dx^2}+\frac{\ell(\ell+1)}{x^2}+x^{2M}-E\right]y(x,E,\ell)=0$$

- large, real positive x asymptotics: $y \sim \frac{x^{-\frac{M}{2}}}{\sqrt{2i}}\exp\left(-\frac{x^{M+1}}{M+1}\right)$
  subdominant (small) solution in the sector $|\arg x| < \frac{\pi}{2M+2}$
- $\boxed{y_k(x,E,\ell)=\omega^{\frac{k}{2}}y(\omega^{-k}x,\omega^{2k}E,\ell)}$ $(\omega=\exp(\frac{2\pi i}{2M+2}))$
  $\{y_k, y_{k+1}\}$ forms a basis of solutions for the ODE:
  - y_k obeys the Stokes relation

$$C(E,\ell)y_0(x,E,\ell)=y_{-1}(x,E,\ell)+y_1(x,E,\ell)$$

The coeffiecient $C(E,\ell)$ is called the Stokes multiplier.



- small x asymptotics: $\psi(x,E,\ell) \sim x^{\ell+1}$ (other solution is $x^{-\ell}$)
- Take the Wronskian ($W[f,g] := fg' - f'g$) of both sides of the Stokes relation with ψ

$$C(E,\ell)W[y_0,\psi](E,\ell) = W[y_{-1},\psi](E,\ell) + W[y_1,\psi](E,\ell)$$

Setting $D(E,\ell) = W[y_0,\psi]$, the above relation is

$$C(E,\ell)D(E,\ell) = \omega^{-(\ell+\frac{1}{2})}D(\omega^{-2}E,\ell) + \omega^{\ell+\frac{1}{2}}D(\omega^2 E,\ell)$$

T-Q relation: (D: Q-function (spectral determinant), C: T-function)

- $\psi_+ = \psi(x,E,\ell)$, $\psi_- = \psi(x,E,-\ell-1)$ are linearly independent solutions. The Wronskian $W[\psi_+,\psi_-]$ yields the quantum Wronskian relations for D.
$(2\ell+1) = \omega^{-(\ell+\frac{1}{2})}D_-(\omega^{-1}E)D_+(\omega E) - \omega^{\ell+\frac{1}{2}}D_-(\omega E)D_+(\omega^{-1}E)$

- One can then derive the Bethe ansatz equation from the T-Q or quantum Wronskian relation.
$E_n^{\pm}$: zeros of $D^{\pm}(E)$

$$\frac{D_{\pm}(\omega^{-2}E_n^{\pm})}{D_{\pm}(\omega^{2}E_n^{\pm})} = -\omega^{\pm 2(\ell+\frac{1}{2})}$$

Expanding $D_{\pm}(E) \sim \prod_{m=1}^{\infty}(1-\frac{E}{E_m^{\pm}})$, we get the BA eq.

$$\prod_{m=1}^{\infty}\frac{E_m^{\pm}-\omega^{-2}E_n^{\pm}}{E_m^{\pm}-\omega^{2}E_n^{\pm}} = -\omega^{\pm(2\ell+1)}$$

Dictionary

ODE	I(ntegrable) M(odel)
$\left[-\frac{d^2}{dx^2}+\frac{\ell(\ell+1)}{x^2}+x^{2M}-E\right]y=0$	6-vertex model
energy E	spectral parameter
degree of potential M	anisotropy
angular momentum ℓ	twist paramter
Stokes multiplier $C(E,\ell)$	Transfer matrix (T-function)
spectral determinant $D(E,\ell)$	Q-operator
the Stokes relation	T-Q relation

modified sinh-Gordon equation: $A_1^{(1)}$

We discuss the relation between the ODE and the modified Sinh-Gordon equation [Lukyanov-Zamolodchikov 1003.5333]

$$\partial_z \partial_{\bar{z}} \eta - e^{2\eta} + p(z)\bar{p}(\bar{z}) e^{-2\eta} = 0, \quad p(z) = z^{2M} - s^{2M}$$

zero curvature condition $[\partial + A, \bar{\partial} + \bar{A}] = 0$

$$\begin{aligned} A &= \frac{1}{2}\partial_z \eta \sigma^3 - e^{\theta}(\sigma^+ e^{\eta} + \sigma^- p e^{-\eta}) \\ \bar{A} &= -\frac{1}{2}\partial_{\bar{z}} \eta \sigma^3 - e^{-\theta}(\sigma^+ e^{\eta} + \sigma^- \bar{p} e^{-\eta}) \end{aligned}$$

asymptotic behavior of $\eta(z, \bar{z})$ at $\rho \to 0, \infty$ ($z = \rho e^{i\phi}$)

- $\eta \to M \log \rho$ ($\rho \to \infty$)
- $\eta \to \ell \log \rho$ ($\rho \to 0$)

We introduce a new parameter ℓ for the boundary condition at $\rho = 0$.

linear system and its solutions

- linear problem $(\partial + A)\Psi = (\bar{\partial} + \bar{A})\Psi = 0$
- linear problem is invariant under
  Ω: $\phi \to \phi + \frac{\pi}{M}$, $\theta \to \theta - \frac{i\pi}{M}$
  Π: $\theta \to \theta + i\pi$, $\hat{\Pi}[A] = \sigma^3 A \sigma^3$
- $\rho \to \infty$, from the WKB analysis, subdominant solution is

$$\Xi \sim \begin{pmatrix} e^{\frac{iM\phi}{2}} \\ e^{-\frac{iM\phi}{2}} \end{pmatrix} \exp\left(-\frac{2\rho^{M+1}}{M+1} \cosh(\theta + i(M+1)\phi) \right)$$

- $\rho \to 0$ basis $\Psi_+(\rho, \phi|\theta) \to \begin{pmatrix} 0 \\ e^{(i\phi+\theta)\ell} \end{pmatrix}$, $\Psi_-(\rho, \phi|\theta) \to \begin{pmatrix} e^{(i\phi+\theta)\ell} \\ 0 \end{pmatrix}$
- $$\Xi = Q_-(\theta)\Psi_+ + Q_+(\theta)\Psi_-$$

  $Q_\pm(\theta)$ are the Q-function of the quantum Sinh-Gordon model

From MShG to ODE

- take the light-cone limit $\bar{z} \to 0$. Then linear system reduced to a differential equation.

$$\Psi = \begin{pmatrix} e^{\frac{\theta}{2}} e^{\frac{\eta}{2}} \psi \\ e^{\frac{-\eta}{2}} e^{\frac{\theta}{2}} (\partial_z + \partial_z \eta) \psi \end{pmatrix}$$

$$\left[\partial_z^2 - u - e^{\theta} p\right] \psi = 0, \quad u = (\partial_z \eta)^2 - \partial_z^2 \eta$$

- conformal limit: $z \to 0$, $\theta \to \infty$

$$x = z e^{\frac{\theta}{M+1}}, \quad E = s^{2M} e^{\frac{2\theta M}{1+M}}, \quad \text{fixed}$$

$$\left[-\partial_x^2 + \frac{\ell(\ell+1)}{x^2} + x^{2M}\right] \psi = E \psi$$

Schrödinger type ODE [Dorey-Tateo, BLZ]

affine Toda field equations (1)

$\mathfrak{g}$: a simple Lie algebra of rank r

$$[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}, \quad \text{for } \alpha+\beta \neq 0,$$
$$[E_\alpha, E_{-\alpha}] = \frac{2\alpha \cdot H}{\alpha^2},$$
$$[H^i, E_\alpha] = \alpha^i E_\alpha.$$

$\alpha_1, \cdots, \alpha_r$: the simple roots of $\mathfrak{g}$
$\alpha_1^\vee, \cdots, \alpha_r^\vee$: simple coroots
$\alpha_0 = -\theta$ (θ: the highest root)
(dual) Coxter labels: $\sum_{i=0}^r n_i \alpha_i = \sum_{i=0}^r n_i^\vee \alpha_i^\vee = 0$.
(dual) Coxter number$h$, $h^\vee$:

$$h = \sum_{i=0}^r n_i, \quad h^\vee = \sum_{i=0}^r n_i^\vee.$$

affine Toda field equations (2)

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\cdot\partial_\mu\phi - \left(\frac{m}{\beta}\right)^2\sum_{i=0}^{r} n_i\left[\exp(\beta\alpha_i\cdot\phi)-1\right],$$

$$\partial^\mu\partial_\mu\phi + \left(\frac{m^2}{\beta}\right)\sum_{i=0}^{r} n_i\alpha_i\exp(\beta\alpha_i\phi) = 0\,.$$

complex coordinates: $z=\frac{1}{2}(x^0+ix^1),\quad \bar{z}=\frac{1}{2}(x^0-ix^1)$ $(z=\rho e^{i\theta})$
conformal transformation ($\rho^\vee$: co-Weyl vector)

$$z\to\tilde{z}=f(z)\,,\quad \phi\to\tilde{\phi}=\phi-\frac{1}{\beta}\rho^\vee\log(\partial f\bar{\partial}\bar{f})\,,$$

modified affine Toda equations:

$$\partial\bar{\partial}\phi + \left(\frac{m^2}{\beta}\right)\left[\sum_{i=1}^{r} n_i\alpha_i\exp(\beta\alpha_i\phi) + p(z)\bar{p}(\bar{z})n_0\alpha_0\exp(\beta\alpha_0\phi)\right] = 0,$$

$$p(z)=(\partial f)^h,\quad \bar{p}(\bar{z})=(\bar{\partial}\bar{f})^h.$$

Lax formalism

- The modified affine Toda equation can be expressed as a linear problem: $(\partial + A)\Psi = 0$ and $(\bar{\partial} + \bar{A})\Psi = 0$.

$$A = \frac{\beta}{2}\partial\phi \cdot H + me^{\lambda}\left\{\sum_{i=1}^{r}\sqrt{n_i^{\vee}}E_{\alpha_i}e^{\frac{\beta}{2}\alpha_i\phi} + p(z)\sqrt{n_0^{\vee}}E_{\alpha_0}e^{\frac{\beta}{2}\alpha_0\phi}\right\},$$

$$\bar{A} = -\frac{\beta}{2}\bar{\partial}\phi \cdot H - me^{-\lambda}\left\{\sum_{i=1}^{r}\sqrt{n_i^{\vee}}E_{-\alpha_i}e^{\frac{\beta}{2}\alpha_i\phi} + \bar{p}(\bar{z})\sqrt{n_0^{\vee}}E_{-\alpha_0}e^{\frac{\beta}{2}\alpha_0\phi}\right\}$$

- zero-curvature condition: $[\partial + A, \bar{\partial} + \bar{A}] = 0 \Longrightarrow$ affine Toda field equations

symmetries and $p(z)$

- Motivated by the ODE/IM correspondence, we put

$$p(z) = z^{hM} - s^{hM}, \quad \bar{p}(\bar{z}) = \bar{z}^{hM} - \bar{s}^{hM}$$

  h: the Coxeter number, and M is some positive real parameter

- We define the transformation $\hat{\Omega}_k$ and $\hat{\Pi}$

$$\hat{\Omega}_k : \begin{cases} z \to z e^{\frac{2\pi k i}{hM}} \\ s \to s e^{\frac{2\pi k i}{hM}} \\ \lambda \to \lambda - \frac{2\pi k i}{hM} \end{cases}$$

$$\hat{\Pi} : \begin{cases} \lambda \to \lambda - \frac{2\pi i}{h} \\ A \to S A S^{-1}, \quad S = \exp(\frac{2\pi i}{h} \rho^{\vee} \cdot H) \end{cases}$$

- The equation of motion and linear problem are invarinant under $\hat{\Omega}_k$ for integer k.
- $S E_{\alpha_i} S^{-1} = e^{2\pi i/h} E_{\alpha_i}$ $(i = 1, \cdots, r)$

asymptotic behavior of the Toda field

- In the large $|z|$ region, asymptotic solution is

$$\phi(z,\bar{z}) = \frac{M}{\beta}\rho^\vee \log(z\bar{z}) + O(1)$$

- For small $|z|$, we assume logarithmic behavior, with expansion

$$\phi(z,\bar{z}) = g\log(z\bar{z}) + \phi^{(0)}(g) + \gamma(z,\bar{z},g) + \sum_{i=0}^{r} \frac{C_i(g)}{(c_i(g)+1)^2}(\bar{z}z)^{c_i(g)+1} + \cdots .$$

  g is an r-component vector

- Substituting this expansion into the Toda equation, we can determine the constants C_i
- The exponents are found to be $c_i + 1 = 1 + \beta\alpha_i \cdot g > 0$.

$A_r^{(1)}$ modified affine Toda [KI-Locke, Adamopuolou-Dunning]

- $A_r^{(1)}$ is the simplest algebra to start with, and includes the sinh-Gordon model as a specific example
- the fundamental representation with highest weight ω_1 weights are $h_1 = \omega_1$, $h_i = \omega_i - \omega_{i+1}$, $h_{r+1} = -\omega_r$, where ω_i are the fundamental weights defined by $\omega_i \cdot \alpha_j^\vee = \delta_{ij}$
- The linear problem $(\partial_z + A)\Psi = 0$, $\Psi = {}^t(\psi_1, \cdots, \psi_{r+1})$

holomorphic connection:

$$A = \begin{pmatrix} \frac{\beta}{2} h_1 \cdot \partial\phi & m e^\lambda e^{\frac{\beta}{2}\alpha_1 \cdot \phi} & 0 & \cdots & 0 \\ 0 & \frac{\beta}{2} h_2 \cdot \partial\phi & m e^\lambda e^{\frac{\beta}{2}\alpha_2 \cdot \phi} & & \vdots \\ & & \ddots & & \\ \vdots & & & \frac{\beta}{2} h_r \cdot \partial\phi & m e^\lambda e^{\frac{\beta}{2}\alpha_r \cdot \phi} \\ m e^\lambda p(z) e^{\frac{\beta}{2}\alpha_0 \cdot \phi} & \cdots & & 0 & \frac{\beta}{2} h_{r+1} \cdot \partial\phi \end{pmatrix}.$$

- gauge transformation: $U = diag(e^{-\frac{\beta}{2}h_1\cdot\phi}, \cdots, e^{-\frac{\beta}{2}h_{r+1}\cdot\phi})$

$$\tilde{A} = UAU^{-1} + U\partial U^{-1}, \quad \tilde{\Psi} = U\Psi,$$

$$\tilde{A} = \begin{pmatrix} \beta h_1\partial\phi & me^{\lambda} & 0 & \cdots & 0 \\ 0 & \beta h_2\partial\phi & me^{\lambda} & & \vdots \\ & & \ddots & & \\ \vdots & & & \beta h_r\partial\phi & me^{\lambda} \\ me^{\lambda}p(z) & & & 0 & \beta h_{r+1}\partial\phi \end{pmatrix}.$$

- the linear problem becomes a single $(r+1)$-th order differential equation

$$D(h_{r+1})\cdots D(h_1)\tilde{\psi}_1 = (-me^{\lambda})^h p(z)\tilde{\psi}_1.$$

$D(h) \equiv \partial + \beta h\cdot\partial\phi$

- scalar Lax operator (Gelfand-Dickii, Drinfeld-Sokolov reduction)

- For the barred linear equation, a different gauge transformation is used to simplify the equations

$$U = diag(e^{\frac{\beta}{2}h_1\cdot\phi}, \cdots, e^{\frac{\beta}{2}h_{r+1}\cdot\phi}), \quad \tilde{\tilde{\Psi}} = U\Psi$$

- The full lienar problem gives the differential equations

$$D(h_{r+1})\cdots D(h_1)\psi = (-me^{\lambda})^h p(z)\psi$$
$$\bar{D}(-h_1)\cdots \bar{D}(-h_{r+1})\bar{\psi} = (me^{-\lambda})^h \bar{p}(\bar{z})\bar{\psi}$$

where $\psi = \tilde{\psi}_1$ and $\bar{\psi} = \tilde{\tilde{\psi}}_{r+1}$

Massive ODE/IM correspondence

- For small $|z|$ solution $\psi^{(i)} \sim z^{\mu_i}$ define the vector $\Psi^{(i)}$ with

$$(\Psi^{(i)})_j \sim \delta_{ij}(\bar{z}/z)^{\frac{\beta}{2}h_i\cdot g}.$$

- For large $|z|$ the small solution is

$$\Xi(\rho,\theta|\lambda) \sim C \begin{pmatrix} e^{-\frac{irM\theta}{4}} \\ e^{-\frac{i(r-2)M\theta}{4}} \\ \vdots \\ e^{\frac{irM\theta}{4}} \end{pmatrix} \exp\left(-\frac{2\rho^{M+1}}{M+1}\, m\cosh(\lambda + i\theta(M+1))\right)$$

- we can expand Ξ as

$$\Xi = \sum_{i=0}^{r} Q_i(\lambda)\Psi^{(i)}.$$

For $A_1^{(1)}$ (sinh-Gordon) $A_2^{(2)}$ (Tzitzéica-Bullough-Dodd), the Q-coefficients correspond to the Q-function of a 2D massive QFT.
$A_r^{(1)}$: KI-Locke, Adamopoulou-Dunning, 1401.1187

Conformal Limit and ODE/IM correspondence

- First we take the light-cone limit $\bar{z} \to 0$ and we define the conformal limit $z \to 0$, $\lambda \to \infty$ with fixed

$$x = (me^{\lambda})^{1/(M+1)} z, \quad E = s^{hM} (me^{\lambda})^{hM/(M+1)}$$

- The differential equation becomes

$$\left[D_x(h_{r+1}) \cdots D_x(h_1) - (-1)^h p(x, E) \right] \psi(x, E, g) = 0$$

  where $D_x(a) = \partial_x + \beta \frac{a \cdot g}{x}$ and $p(x, E) \equiv x^{hM} - E$.
- This is the ODE for A_r-type Lie algebra Suzuki, Dorey-Dunning-Tateo
- By writing out the unique asymptotically decaying solution $\xi(x, E, g)$ to this equation in terms of the small x basis $\chi^{(i)} \sim x^{\mu_i} + \mathcal{O}(x^{\mu_i + h})$, we have $\xi(x, E, g) = \sum_{i=0}^{r} Q^{(i)}(E) \chi^{(i)}(x, E, g)$

- Symanzik rotation $\psi_k(x,E,g)=\psi(\omega^k x,\Omega^k E,g)$ with $\Omega=\exp(i\frac{2\pi M}{M+1})$ and $\omega=\exp(i\frac{2\pi}{h})$
- auxiliary functions: $\psi^{(a)}=W^{(a)}[\psi_{\frac{1-a}{2}},\cdots,\psi_{\frac{a-1}{2}}]$ $(a=2,\cdots,r)$
- A_n ψ-system (Plücker relations)

$$\psi^{(a-1)}\psi^{(a+1)}=W[\psi^{(a)}_{-\frac{1}{2}},\psi^{(a)}_{\frac{1}{2}}],\quad \psi^{(0)}=\psi^{(n)}=1$$

- quantum Wronskian relation

$$Q^{(a+1)}Q^{(a-1)}=\omega^{\frac{1}{2}(\mu_a-\mu_{a-1})}Q^{(a)}_{-\frac{1}{2}}\bar{Q}^{(a)}_{\frac{1}{2}}-\omega^{\frac{1}{2}(\mu_{a-1}-\mu_a)}Q^{(a)}_{\frac{1}{2}}\bar{Q}^{(a)}_{-\frac{1}{2}}$$

- Bethe ansatz equation

$$\omega^{\mu_{i-1}-\mu_i}\frac{Q^{(i-1)}_{-1/2}(E^{(i)}_n)Q^{(i)}_1(E^{(i)}_n)Q^{(i+1)}_{-1/2}(E^{(i)}_n)}{Q^{(i-1)}_{1/2}(E^{(i)}_n)Q^{(i)}_{-1}(E^{(i)}_n)Q^{(i+1)}_{1/2}(E^{(i)}_n)}=-1\,.$$

where $E^{(i)}_n$ are zeros of $Q^{(i)}(E)$.

Other affine Lie algberas

[KI-Locke,1312.6759]

- We will consider the other affine Lie algebras and find the (pseudo-)diffrential equations associated to the linear problem for the fundamental representation.

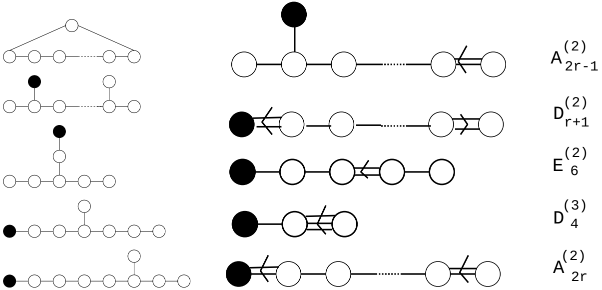
$A_r^{(1)}$	$D(\mathbf{h})\psi = (-me^{\lambda})^h p(z)\psi$
$D_r^{(1)}$	$D(\mathbf{h}^\dagger)\partial^{-1}D(\mathbf{h})\psi = 2^{r-1}(me^{\lambda})^h\sqrt{p(z)}\partial\sqrt{p(z)}\psi$
$B_r^{(1)}$	$D(\mathbf{h}^\dagger)\partial D(\mathbf{h})\psi = 2^{r}(me^{\lambda})^h\sqrt{p(z)}\partial\sqrt{p(z)}\psi$
$A_{2r-1}^{(2)}$	$D(\mathbf{h}^\dagger)D(\mathbf{h})\psi = -2^{r-1}(me^{\lambda})^h\sqrt{p(z)}\partial\sqrt{p(z)}\psi$
$C_r^{(1)}$	$D(\mathbf{h}^\dagger)D(\mathbf{h})\psi = (me^{\lambda})^h p(z)\psi$
$D_{r+1}^{(2)}$	$D(\mathbf{h}^\dagger)\partial D(\mathbf{h})\psi = 2^{r+1}(me^{\lambda})^{2h}p(z)\partial^{-1}p(z)\psi$
$A_{2r}^{(2)}$	$D(\mathbf{h}^\dagger)\partial D(\mathbf{h})\psi = -2^{r}\sqrt{2}(me^{\lambda})^h p(z)\psi$
$G_2^{(1)}$	$D(\mathbf{h}^\dagger)\partial D(\mathbf{h})\psi = 8(me^{\lambda})^h\sqrt{p(z)}\partial\sqrt{p(z)}\psi$
$D_4^{(3)}$	$D(\mathbf{h}^\dagger)\partial D(\mathbf{h})\psi + (\omega+1)2\sqrt{3}(me^{\lambda})^4 D(\mathbf{h}^\dagger)p(z)$ $-(\omega+1)2\sqrt{3}(me^{\lambda})^4 pD(\mathbf{h}) - 8\sqrt{3}\omega(me^{\lambda})^3 D(-h_1)\sqrt{p}\partial\sqrt{p}D(h_1)$ $+(\omega-1)^3 12(me^{\lambda})^8 p\partial^{-1}p\}\psi = 0$

$D(\mathbf{h}) = D(h_r)\cdots D(h_1)$, $D(\mathbf{h}^\dagger) = D(-h_1)\cdots D(-h_r)$ for $\mathbf{h} = (h_r,\cdots,h_1)$

Langlands duality

Langlands (GNO) dual: $\hat{\mathfrak{g}}$: simple roots $\alpha_i \Longleftrightarrow \hat{\mathfrak{g}}^\vee$: $\alpha_i^\vee$ simple coroots

- $\hat{\mathfrak{g}}^\vee = X_r^{(1)}$ for $X = ADE$
- $\hat{\mathfrak{g}}^\vee = X_r^{(s)}$ for non-simply laced $\hat{\mathfrak{g}}$ (twisted affine Lie algebra) $(B_r^{(1)})^\vee = A_{2r-1}^{(2)}$, $(C_r^{(1)})^\vee = D_{r+1}^{(2)}$, $(F_4^{(1)})^\vee = E_6^{(2)}$, $(G_2^{(1)})^\vee = D_4^{(3)}$, $(A_{2r}^{(2)})^\vee = A_{2r}^{(2)}$



Langlands duality(2)

- In Dorey-Dunning-Masoero-Suzuki-Tateo (2007), they found a set of pseudo-differential equations associated to classical Lie algebras

affine Toda equation	ODE(Dorey et al.)
$A_r^{(1)}$	A_r
$(B_r^{(1)})^\vee = A_{2r-1}^{(2)}$	B_r
$(C_r^{(1)})^\vee = D_{r+1}^{(2)}$	C_r
$D_r^{(1)}$	D_r

ψ-system and Bethe ansatz equations

We want to prove that the modified affine Toda equation for the Langlands dual $\hat{\mathfrak{g}}^\vee$ corresponds to the $\mathfrak{g}$-type Bethe ansatz equation.

- ODE is complicated for higher-rank affine Lie algebras (E-type ODE?)
- One can derive the ψ-system based on the linear system (classical Lie algebra in the conformal limit [Sun,1201.1614])

We consider the affine Toda field equations for $\hat{\mathfrak{g}}$

- Applying the gauge transformation

$$U_A = z^{M\rho^\vee \cdot H} e^{-\beta\phi \cdot H/2}$$

  gives a simple form of the linear problem in the large ρ limit,

$$\tilde{A} = m e^{\lambda} z^M \Lambda_+, \quad \tilde{\bar{A}} = m e^{-\lambda} \bar{z}^M \Lambda_-$$

$$\Lambda_\pm = \sqrt{n_0^\vee} E_{\pm\alpha_0} + \sum_{i=1}^{r} \sqrt{n_i^\vee} E_{\pm\alpha_i}$$

- We will consider the fundamental representations $V^{(a)}$ with the highest weight ω_a $(a = 1, \cdots, r)$ of $\hat{\mathfrak{g}}$.
  $\mathbf{e}_i^{(a)}$: a basis of $V^{(a)}$ and $\mathbf{e}_1^{(a)}$ is the highest weight vector
- asymptotic form for a subdominant solutions along the positive real axis

$$\Psi^{(a)}(z, \bar{z}|\lambda) = \exp\left(-2\mu^{(a)} \frac{\rho^{M+1}}{M+1} m \cosh\left(\lambda + i\theta(M+1)\right)\right) e^{-i\theta M \rho^{\vee} \cdot H} \boldsymbol{\mu}^{(a)}.$$

$\mu^{(a)}$ and $\boldsymbol{\mu}^{(a)}$ are the eigenvalues of $\Lambda_+^{(a)} = (\Lambda_-^{(a)})^T$ with the largest real part and its eigenvector in module $V^{(a)}$.

- For small z,

$$\Psi^{(a)}(z, \bar{z}|\lambda, g) = \sum_{i=1}^{\dim(V^{(a)})} Q_i^{(a)}(\lambda, g)\ \mathcal{X}_i^{(a)}(z, \bar{z}|\lambda, g)$$

$$\mathcal{X}_i^{(a)} = e^{-(\lambda + i\theta)\beta g \cdot h_i^{(a)}} \mathbf{e}_i^{(a)} + \mathcal{O}(\rho) \ \text{ as } \ \rho \to 0$$

ψ-system

- embedding map:

$$\iota : \bigwedge^{2} V^{(a)} \rightarrow \bigotimes_{b=1}^{r} \left(V^{(b)}\right)^{B_{ab}} .$$

- highest weight: $2\omega_a - \alpha_a = \sum_{b=1}^{r} B_{ab}\omega_b$
  A_{ab}: Cartan matrix of $\mathfrak{g}$ and B_{ab} the incidence matrix

$$B_{ab} = 2\delta_{ab} - A_{ab} .$$

- take the anti-symmetric product of for $\Psi^{(a)}_{\pm 1/2}$ and decompose it by the embedding map such that large ρ asymptotics matches
- the largest eigenvalues $\mu^{(a)}$ of $\Lambda^{(a)}_{+}$ constrained such that the two asymptotics coincide with each other

ψ-system for $\hat{\mathfrak{g}}^\vee$

$\hat{\mathfrak{g}} = A, D, E$ [Sun, Masoero-Raimondo-Valeri, KI-Locke]

- ψ-system

$$\iota\left(\Psi^{(a)}_{-1/2}\wedge\Psi^{(a)}_{1/2}\right)=\bigotimes_{b=1}^{r}\left(\Psi^{(b)}\right)^{B_{ab}}.$$

- $\mu^{(a)}$ satisfy the equations

$$2\cos\left(\pi/h\right)\mu^{(a)}=\sum_{b=1}^{r}B_{ab}\,\mu^{(b)},$$

- These eigenvalues coincide with the mass spectrum of affine Toda field theories for $\hat{\mathfrak{g}}$ [Braden-Corrigan-Dorey-Sasaki]

Eigenvalues $\mu^{(a)}$ of $\Lambda^{(a)}_{+}$ $\Longleftrightarrow$ Eigenvalues of $(m^2)^{ab}=\sum_{i=0}^{r}\alpha_i^a\alpha_i^b$

For $A^{(1)}_r$, $\mu^{(a)}=\sin\frac{\pi a}{r+1}/\sin\frac{\pi}{r+1}$

ψ-system for $\hat{\mathfrak{g}}^\vee$ ($\hat{\mathfrak{g}} = B_r^{(1)}, C_r^{(1)}, F_4^{(1)}, G_2^{(1)}$)

$$(B_r^{(1)})^\vee = A_{2r-1}^{(2)}: \quad \iota\left(\Psi_{-1/2}^{(a)} \wedge \Psi_{1/2}^{(a)}\right) = \Psi^{(a-1)} \otimes \Psi^{(a+1)} \quad \text{for } a = 1, \ldots, r-1,$$

$$\iota\left(\Psi_{-1/4}^{(r)} \wedge \Psi_{1/4}^{(r)}\right) = \Psi_{-1/4}^{(r-1)} \otimes \Psi_{1/4}^{(r-1)}.$$

$$(C_r^{(1)})^\vee = D_{r+1}^{(2)}: \quad \iota\left(\Psi_{-1/4}^{(a)} \wedge \Psi_{1/4}^{(a)}\right) = \Psi^{(a-1)} \otimes \Psi^{(a+1)} \quad \text{for } a = 1, \ldots, r-2,$$

$$\iota\left(\Psi_{-1/4}^{(r-1)} \wedge \Psi_{1/4}^{(r-1)}\right) = \Psi^{(r-2)} \otimes \Psi_{-1/4}^{(r)} \otimes \Psi_{1/4}^{(r)},$$

$$\iota\left(\Psi_{-1/2}^{(r)} \wedge \Psi_{1/2}^{(r)}\right) = \Psi^{(r-1)}.$$

$$(F_4^{(1)})^\vee = E_6^{(2)}: \quad \iota\left(\Psi_{-1/2}^{(1)} \wedge \Psi_{1/2}^{(1)}\right) = \Psi^{(2)}, \quad \iota\left(\Psi_{-1/2}^{(2)} \wedge \Psi_{1/2}^{(2)}\right) = \Psi^{(1)} \otimes \Psi^{(3)},$$

$$\iota\left(\Psi_{-1/4}^{(3)} \wedge \Psi_{1/4}^{(3)}\right) = \Psi^{(2)} \otimes \Psi_{-1/4}^{(4)} \otimes \Psi_{1/4}^{(4)}, \quad \iota\left(\Psi_{-1/4}^{(4)} \wedge \Psi_{1/4}^{(4)}\right) = \Psi^{(3)}.$$

$$(G_2^{(1)})^\vee = D_4^{(3)}: \quad \iota\left(\Psi_{1/2}^{(1)} \wedge \Psi_{1/2}^{(1)}\right) = \Psi^{(2)},$$

$$\iota\left(\Psi_{1/6}^{(2)} \wedge \Psi_{1/6}^{(2)}\right) = \Psi_{-2/6}^{(1)} \otimes \Psi_{0}^{(1)} \otimes \Psi_{2/6}^{(1)}.$$

Comments

- $\hat{\mathfrak{g}} = B_r^{(1)}, C_r^{(1)}$ [Sun, 1201.1614] (massless limit)
- For $\hat{\mathfrak{g}} = F_4^{(1)}, G_2^{(1)}$, the ψ-system coincides with that conjectured by [Dorey-Dunning-Masoero-Suzuki-Tateo, 0612298]
- The eigenvalues $\mu^{(a)}$ do not coincide with those of mass matrix of affine Toda field theories.
  - $A_{2r-1}^{(2)}$: $\mu^{(a)} = \frac{\sqrt{2}}{\sin\frac{\pi}{2r-1}} \sin\frac{\pi a}{2r-1}$
    Affine Toda field theory:
    $m_a = 2\sqrt{2} m \sin\frac{a\pi}{2r-1} (a = 1, \cdots, r-1), \quad m_r = \sqrt{2} m$
  - $D_4^{(3)}$: $\mu^{(2)}/\mu^{(1)} = \sqrt{2}$
    AFT: $\mu^{(2)}/\mu^{(1)} = \sqrt{\frac{3+\sqrt{3}}{3-\sqrt{3}}}$

Bethe-ansatz equations for affine Toda field equations

- conformal limit We first take the light-cone limit $\bar{z} \to 0$. Then consider the limit $\lambda \to \infty$ and $z, s \to 0$ with fixed

$$x = (me^{\lambda})^{1/(M+1)} z, \quad E = s^{hM} (me^{\lambda})^{hM/(M+1)} ,$$

- the solution $\Psi^{(a)}$ becomes

$$\psi^{(a)}(x, E) = Q^{(a)}(E)\, \chi_1^{(a)}(x, E) + \tilde{Q}^{(a)}(E)\, \chi_2^{(a)}(x, E) + \cdots ,$$

  with $\chi_i^{(a)} \sim x^{\lambda_i^{(a)}}$

- Substituting the ψ-system we obtain the quantum Wronskian relations. For $A_r^{(1)}$ case, for example, we find that

$$\omega^{-\frac{1}{2}(\lambda_1^{(a)} - \lambda_2^{(a)})} Q_{-1/2}^{(a)} \tilde{Q}_{1/2}^{(a)} - \omega^{\frac{1}{2}(\lambda_1^{(a)} - \lambda_2^{(a)})} Q_{1/2}^{(a)} \tilde{Q}_{-1/2}^{(a)} = Q^{(a-1)} Q^{(a+1)}.$$

  where $Q_k^{(a)}(E) \equiv Q^{(a)}(\omega^{hMk} E)$ $(\omega = \exp(2\pi i / h(M+1)))$

- Let us denote the zeros of $Q^{(a)}(E)$ as $E_k^{(a)}$. Then substituting the $\pm 1/2$ Symanzik rotation of the quantum Wronskian relations yields the Bethe ansatz equations.

$A_r^{(1)}$, $D_r^{(1)}$, $E_r^{(1)}$:

$$\prod_{b=1}^{r} \frac{Q^{(b)}_{A_{ab}/2}}{Q^{(b)}_{-A_{ab}/2}}\Bigg|_{E_k^{(a)}} = -\omega^{1+\beta\alpha_a\cdot g}.$$

$(B_r^{(1)})^\vee = A_{2r-1}^{(2)}$:

$$\frac{Q^{(a-1)}_{-1/2} Q^{(a)}_{1} Q^{(a+1)}_{-1/2}}{Q^{(a-1)}_{1/2} Q^{(a)}_{-1} Q^{(a+1)}_{1/2}}\Bigg|_{E_i^{(a)}} = -\omega^{1+\beta\alpha_a\cdot g} \quad \text{for } a = 1,\ldots,r-1,$$

$$\frac{Q^{(r-1)}_{-1/2} Q^{(r)}_{1/2}}{Q^{(r-1)}_{1/2} Q^{(r)}_{-1/2}}\Bigg|_{E_i^{(r)}} = -\omega^{\frac{1}{2}(1+\beta\alpha_r\cdot g)}.$$

$(C_r^{(1)})^\vee = D_{r+1}^{(2)}$:

$$\frac{Q^{(a-1)}_{-1/4} Q^{(a)}_{1/2} Q^{(a+1)}_{-1/4}}{Q^{(a-1)}_{1/4} Q^{(a)}_{-1/2} Q^{(a+1)}_{1/4}}\Bigg|_{E_i^{(a)}} = -\omega^{\frac{1}{2}(1+\beta\alpha_a\cdot g)} \quad \text{for } a = 1,\ldots,r-2,$$

$$\frac{Q^{(r-2)}_{-1/4} Q^{(r-1)}_{1/2} Q^{(r)}_{-1/2}}{Q^{(r-2)}_{1/4} Q^{(r-1)}_{-1/2} Q^{(r)}_{1/2}}\Bigg|_{E_i^{(r-1)}} = -\omega^{\frac{1}{2}(1+\beta\alpha_{r-1}\cdot g)}, \quad \frac{Q^{(r-1)}_{-1/2} Q^{(r)}_{1}}{Q^{(r-1)}_{1/2} Q^{(r)}_{-1}}\Bigg|_{E_i^{(r)}} = -\omega^{1+\beta\alpha_r\cdot g}.$$

$(F_4^{(1)})^\vee = E_6^{(2)}$:

$$\left.\frac{Q_1^{(1)} Q_{-1/2}^{(2)}}{Q_{-1}^{(1)} Q_{1/2}^{(2)}}\right|_{E_i^{(1)}} = -\omega^{1+\beta\alpha_1\cdot g}, \quad \left.\frac{Q_{-1/2}^{(1)} Q_1^{(2)} Q_{-1/2}^{(3)}}{Q_{1/2}^{(1)} Q_{-1}^{(2)} Q_{1/2}^{(3)}}\right|_{E_i^{(2)}} = -\omega^{1+\beta\alpha_2\cdot g},$$

$$\left.\frac{Q_{-1/4}^{(1)} Q_{1/2}^{(2)} Q_{-1/2}^{(3)}}{Q_{1/4}^{(1)} Q_{-1/2}^{(2)} Q_{1/2}^{(3)}}\right|_{E_i^{(3)}} = -\omega^{\frac{1}{2}(1+\underline{\mathsf{e}}ta\alpha_3\cdot g)}, \quad \left.\frac{Q_{-1/4}^{(3)} Q_{1/2}^{(4)}}{Q_{1/4}^{(3)} Q_{-1/2}^{(4)}}\right|_{E_i^{(4)}} = -\omega^{\frac{1}{2}(1+\beta\alpha_4\cdot g)}.$$

$(G_2^{(1)})^\vee = D_4^{(3)}$:

$$\left.\frac{Q_1^{(1)} Q_{-1/2}^{(2)}}{Q_{-1}^{(1)} Q_{1/2}^{(2)}}\right|_{E_i^{(1)}} = -\omega^{1+\beta\alpha_1\cdot g},$$

$$\left.\frac{Q_{-1/2}^{(1)} Q_{2/6}^{(2)}}{Q_{1/2}^{(1)} Q_{-2/6}^{(2)}}\right|_{E_i^{(2)}} = -\omega^{\frac{1}{3}(1+\beta\alpha_2\cdot g)}.$$

These are the Bethe ansatz equations for $\hat{\mathfrak{g}}$ classified by Reshetikhin and Wiegmann (1987) and Kuniba-Suzuki (1995) based on $U_q(\hat{\mathfrak{g}})$

Twisted affine Lie algebra $A_{2r}^{(2)}$

- ψ-system

$$\iota\left(\Psi^{(a)}_{-1/2}\wedge\Psi^{(a)}_{1/2}\right)=\Psi^{(a-1)}\otimes\Psi^{(a+1)},$$
$$\iota\left(\Psi^{(r)}_{-1/2}\wedge\Psi^{(r)}_{1/2}\right)=\Psi^{(r-1)}\otimes\Psi^{(r)}.$$

- Bethe ansatz equation:

$$\left.\frac{Q^{(a-1)}_{-1/2}Q^{(a)}_{1}Q^{(a+1)}_{-1/2}}{Q^{(a-1)}_{1/2}Q^{(a)}_{-1}Q^{(a+1)}_{1/2}}\right|_{E_i^{(a)}}=-\omega^{1+\beta\alpha_a\cdot g},\quad \left.\frac{Q^{(r-1)}_{-1/2}Q^{(r)}_{-1/2}Q^{(r)}_{1}}{Q^{(r-1)}_{1/2}Q^{(r)}_{1/2}Q^{(r)}_{-1}}\right|_{E_i^{(r)}}=-\omega^{1+\beta\alpha_r\cdot g}.$$

for $a=1,\dots,r-1$

- For $r=1$, Tzitzéica-Bullough-Dodd model $\leftrightarrow$ BA eq. for Izergin-Korepin model [Dorey-Tateo, Dorey-Faldella-Negro-Tateo]
- $r\geq 1$ BA eqs. for $U_q(A_{2r}^{(2)})$ [Reshetikhin-Wiegmann,Kuniba-Suzuki]

ODE or PDE	$\Longleftrightarrow$ ODE/IM	BAE massless TBA	$\Longleftrightarrow$	CFT
$\Uparrow$ Conformal limit	ψ-system	$\Uparrow$ UV limit		
Linear problem	$\Longleftrightarrow$ massive ODE/IM	BAE massive TBA	$\Longleftrightarrow$	massive QFT
$\Uparrow$				
affine Toda equation				

Outlook

- Our dictionary of ODE/IM correspondence is not yet complete
  - general $p(z)$ Bazhanov-Lukyanov
  - ODE/IM for $B_r^{(1)}$, $C_r^{(1)}$, $G_2^{(1)}$, $F_4^{(1)}$
    Bethe-ansatz(-like) equations (IM?)
  - affine Lie superalgebra
  - generalized Drinfeld-Sokolov reduction
    [Balog-Feher-O'Raifeartaigh-Forgacs-Wipf]
  - Gaudin-type Bethe equations [Feigin-Frenkel]
  - T-system, Y-system, nonilinear integral equations
- Why this correspondence holds? Why Langlands dual?
  We need to understand this correspondence in a stringy setup.
  - AdS_4 and AdS_5 minimal surface (null-poly Wilson loop, form factor)
    affine $B_2^{(1)}$ Toda field equation KI-Locke-Satoh-Shu, work in progress
  - gauge/Bethe correspondence [Nekrasov-Shatashvili, ...,
    Chen-Hsin-Koroteev]
  - Langlands dual and Hitchin system [Kapustin-Witten]