4 Magnetic Monopole in Gauge Theories

References:
2. J. Harvey, arXiv:9603042\[hep-th\]

Maxwell equations In the presence of magnetic source and currents, Maxwell’s equations are modified as (in the Heaviside-Lorentz unit)
\[
\nabla \cdot (\mathbf{E} + i\mathbf{B}) = \rho_e + i\rho_m \tag{4.1}
\]
\[
\nabla \times (\mathbf{E} + i\mathbf{B}) - i\partial_t (\mathbf{E} + i\mathbf{B}) = i(\mathbf{j}_e + i\mathbf{j}_m) \tag{4.2}
\]
is invariant under the $U(1)$ transformation:
\[
\mathbf{E} + i\mathbf{B} \rightarrow e^{i\theta} (\mathbf{E} + i\mathbf{B}) \tag{4.3}
\]
\[
\rho_e + i\rho_m \rightarrow e^{i\theta} (\rho_e + i\rho_m) \tag{4.4}
\]
\[
(\mathbf{j}_e + i\mathbf{j}_m) \rightarrow e^{i\theta} (\mathbf{j}_e + i\mathbf{j}_m) \tag{4.5}
\]
For $\theta = -\frac{\pi}{2}$, we have the electromagnetic duality transformation:
\[
\mathbf{E} \rightarrow \mathbf{B}, \quad \rho_e \rightarrow \rho_m, \quad \mathbf{j}_e \rightarrow \mathbf{j}_m \tag{4.6}
\]
Then the Lorentz force per unit volume is modified as
\[
f = \rho_e \mathbf{E} + \mathbf{j}_e \times \mathbf{B} + \rho_m \mathbf{B} - \mathbf{j}_m \times \mathbf{E} \tag{4.7}
\]
In the covariant form, Maxwell equations take the form
\[
\partial_\mu F^{\mu\nu} = j_\nu, \quad \partial_\mu \tilde{F}^{\mu\nu} = j_\nu. \tag{4.8}
\]
Then duality transformation (4.6) reads
\[
F^{\mu\nu} \rightarrow \tilde{F}^{\mu\nu}, \quad j_\nu \rightarrow j_\nu \tag{4.9}
\]
\[
\tilde{F}^{\mu\nu} \rightarrow F^{\mu\nu}, \quad j_\nu \rightarrow -j_\nu \tag{4.10}
\]
where $F^{0i} = -E^i$, $F^{ij} = -\epsilon^{ijk} B^k$, \(j_\mu = (\rho_e, \mathbf{j}_e)\) and \(j_\mu = (\rho_m, \mathbf{j}_m)\). Lorentz force
\[
m \frac{d^2 x^\nu}{d\tau^2} = (qF^{\mu\nu} + g\tilde{F}^{\mu\nu}) \frac{dx_\nu}{d\tau} \tag{4.11}
\]
Wu-Yang monopole The vector potential $A_{\mu}$ cannot be defined as a continuous function of spacetime coordinates. If we write $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, then it follows $\partial_\mu \tilde{F}^{\mu\nu} = 0$.

Dirac monopole A magnetic monopole centered at the origin produces the magnetic field

$$B = \frac{g}{4\pi r^3}. \quad (4.12)$$

The vector potential $A$ satisfying $B = \nabla \times A$ is, for example, given by

$$A = \frac{g}{4\pi r} \frac{1 - \cos \theta}{\sin \theta} e_{\phi}. \quad (4.13)$$

But this potential becomes singular at $\theta = \pi$ and is not defined in the whole space.

Wu-Yang’s idea: The potential should not be singular but the gauge transformation may be discontinuous. Let us consider the 2-sphere surrounding the monopole. The sphere decomposes into the northern hemisphere $S_N$ and the southern hemisphere $S_S$. In $S_N$, the gauge potential is

$$A_N = \frac{g}{4\pi r} \frac{1 - \cos \theta}{\sin \theta} e_{\phi}. \quad (4.14)$$

In $S_S$, the gauge potential is

$$A_S = -\frac{g}{4\pi r} \frac{1 + \cos \theta}{\sin \theta} e_{\phi}. \quad (4.15)$$

At the equator $\theta = \frac{\pi}{2}$, the defined regions of two potentials overlap and the potentials are related by the gauge transformation:

$$A_N - A_S = -\nabla \chi, \quad \chi = -\frac{g}{2\pi} \phi \quad (4.16)$$

and $\chi$ is a discontinuous function along the equator.

Exercise 9.1 Confirm that the gauge potentials (4.14) and (4.15) produce the magnetic field (4.12).

Hint: Use $\nabla = e_r \frac{\partial}{\partial r} + e_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + e_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}$.

The magnetic flux through the sphere is

$$\int B \cdot dS = \int_{equator} d\chi = \chi(0) - \chi(2\pi) = g \quad (4.17)$$
which gives the monopole charge.

Note: This formulation leads to the fact that gauge theory can be defined by the concepts of fibre bundle.

**quantization of electric charge** The Schrödinger eq.

\[ i\hbar \frac{\partial}{\partial t} \psi = -\frac{\hbar^2}{2m} (\nabla - ieA)^2 \psi + e\phi \psi \]  
\[ (4.18) \]

is invariant the gauge transformation

\[ \psi \rightarrow e^{-\frac{i e}{\hbar} \chi} \psi \]  
\[ (4.19) \]

\[ A \rightarrow A - \nabla \chi, \quad \phi \rightarrow \phi + \frac{\partial \chi}{\partial t} \]  
\[ (4.20) \]

In the presence of monopole, the gauge transformation is given by (4.16). Since the wave function must be continuous and single-valued function,

\[ \frac{eg}{\hbar} = 2\pi n \quad n \in \mathbb{Z} \]  
\[ (4.21) \]

**dyon** Dyon is the particle both electric and magnetic charges. If a dyon with charge \((q, g)\) is at the origin, the electric and magnetic fields are given by

\[ E = \frac{q r}{4\pi r^3}, \quad B = \frac{g r}{4\pi r^3} \]  
\[ (4.22) \]

The test dyon particle with charge \((q', g')\) receives a Lorentz force

\[ F = \dot{r} \times (q' B - g' E) \]  
\[ (4.23) \]

Then the angular momentum of the particle

\[ \frac{d}{dt} (mr \times \dot{r}) = \frac{q'g - g'q}{4\pi} \left( \frac{\dot{r}}{r} - \frac{(r \cdot \dot{r})}{r^3} r \right) = \frac{d}{dt} \left( \frac{q'g - g'q}{4\pi} \frac{r^2}{r} \right) \]  
\[ (4.24) \]

Then

\[ J := mr \times \dot{r} - \frac{q'g - g'q}{4\pi} \frac{r^2}{r} \]  
\[ (4.25) \]

is conserved. The second term is the angular momentum of the electro-magnetic field. Since the angular momentum is quantized in quantum theory, we find

\[ q'g - g'q = 2\pi n \hbar \]  
\[ (4.26) \]

(Dirac-Schwinger-Zwanziger quantization condition)
Georgi-Glashow model [3]: \( G = SO(3) \)

\[
\mathcal{L} = -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} + \frac{1}{2} D^{\mu} \phi^a D_{\mu} \phi^a - V(\phi)
\]  

\[
F^{\mu \nu} = \partial_{\mu} A_{\nu}^a - \partial_{\nu} A_{\mu}^a + e \epsilon^{abc} A_{\mu}^b A_{\nu}^c
\]

\[
D_{\mu} \phi^a = \partial_{\mu} \phi^a + e \epsilon^{abc} A_{\mu}^b \phi^c
\]

\[
V(\phi) = \frac{\lambda}{4} (\phi^a \phi^a - v^2)^2
\]

equation of motion

\[
D_{\mu} F^{\mu \nu} = e \epsilon^{abc} \phi^b D^\nu \phi^c
\]

\[
(D^\mu D_\mu \phi)^a = -\lambda \phi^a (\phi^2 - v^2)
\]

Bianchi identity

\[
D_\mu \tilde{F}^{\mu \nu} = 0
\]

energy-momentum tensor:

\[
\Theta^{\mu \nu} = -F^{\rho \mu} F^\rho_{\ \nu} + D_\mu \phi^a D^\nu \phi^a - \eta^{\mu \nu} \mathcal{L}
\]

\( \Theta^{\mu \nu} \) is conserved \( \partial_\mu \Theta^{\mu \nu} = 0 \).

energy density

\[
\Theta_{00} = \frac{1}{2} (E^{ai} E_{ai} + B^{ai} B_{ai} + (D_0 \phi^a)(D_0 \phi^a) + (D_i \phi^a)(D_i \phi^a)) + V(\phi)
\]

\[
E^{ai} = -F^{ahi}, \quad B^{ai} = -\frac{1}{2} \epsilon^{ijk} F^{ajk}
\]

- \( \Theta_{00} \geq 0 \)

- \( \Theta_{00} = 0 \): vacuum=minimum of the energy

\( F^{\mu \nu} = 0 \) (pure gauge), \( D_{\mu} \phi^a = 0 \), \( V(\phi) = 0 \) (vacuum configuration of field = \( S^2 \))
perturbative spectrum of the Georgi-Glashow model  excitation from the vacuum 
\[ \langle \phi^a \rangle \neq 0: \text{spontaneous symmetry breaking} \]

For \( \langle \phi^a \rangle = v \delta^a_3 \), we expand \( \phi^a = \chi^a + \langle \phi^a \rangle \).

\[
V(\phi) = \frac{\lambda}{4} (\chi^a \chi^a + 2 \chi^3 v)^2 = \lambda v^2 (\chi^3)^2 + \cdots \tag{4.37}
\]

\( \chi^1 \) and \( \chi^2 \) are massless and \( \chi^3 \) acquires mass \( \sqrt{2} \sqrt{\lambda} v \).

\[
\frac{1}{2} (D_\mu \phi)^2 = \frac{e^2}{2} (\epsilon^{abc} A^b_\mu (\phi^c))^2 + \cdots \]

\[
= \frac{e^2 a^2}{2} ((A^1_\mu)^2 + (A^2_\mu)^2) + \cdots \tag{4.38}
\]

The gauge bosons \( A^1_\mu, A^2_\mu \) become massive with mass \( m_1 = m_2 = ea \), while \( A^3_\mu \) is kept to be massless. It is the \( U(1) \) gauge field. The gauge symmetry \( SO(3) \) is broken to \( U(1) \): \( SO(3) \to U(1) \).

finite energy configuration  The Georgi-Glashow model has static finite energy solution of the field equations. The solution of the scalar field must be in the vacuum at spatial infinity:

\[ \phi^a \to \langle \phi^a \rangle \quad (r \to \infty) \tag{4.39} \]

and \( \langle \phi^a \rangle^2 = v^2 \), which forms a two-dimensional sphere \( S^2 \). This defines a continuous map from two-dimensional sphere \( S^2_\infty \) at infinity in three-dimensional space:

\[ S^2_\infty \to S^2. \tag{4.40} \]

The continuous map from \( S^2 \) to \( S^2 \) are classified by an integer \( n \in Z \) called the winding number. The homotopy group \( \pi_2(S^2) = Z \).

The boundary condition for \( \phi^a \) is

\[ D_\mu \phi^a := \partial_\mu \phi^a + e \epsilon^{abc} A^b_\mu \phi^c \sim o(\frac{1}{r}) \quad r \to \infty \tag{4.41} \]

The solution of \( D_\mu \phi^a = 0 \) is

\[ A^a_\mu = \frac{1}{v} \phi^a A_\mu - \frac{1}{ev^2} \epsilon^{abc} \phi^b \partial_\mu \phi^c \tag{4.42} \]
where $A_\mu$ is arbitrary. The field strength:

$$F^{\mu\nu} = {1 \over v} \phi^a F^{\mu\nu}$$

(4.43)

where

$$F^{\mu\nu} = -{1 \over ev^3} \epsilon^{abc} \phi^a \partial^\mu \phi^b \partial^\nu \phi^c + \partial^\mu A^\nu - \partial^\nu A^\mu$$

(4.44)

magnetic charge

$$g = \int_{S_\infty} B \cdot dS = {1 \over 2ev^3} \int_{S_\infty} dS^i \epsilon^{ijk} \epsilon^{abc} \phi^a \partial^j \phi^b \partial^k \phi^c = {4\pi N \over e}, \quad N \in \mathbb{Z}$$

(4.45)

EM charge is quantized

$$eg = 4\pi N$$

(4.46)

**Exercise 10.1** For $\phi^a \to v {\xi^a \over r}$ ($r \to \infty$) show that

$$N := {1 \over 8\pi v^3} \int_{S_\infty} dS^i \epsilon^{ijk} \epsilon^{abc} \phi^a \partial^j \phi^b \partial^k \phi^c = 1$$

(4.47)

't Hooft-Polyakov monopole ansatz of the solution[4, 5]:

$$\phi^a = {r^a \over er^2} H(\xi)$$

(4.48)

$$A^a_i = \epsilon^{aji} {r^j \over er^2} (1 - K(\xi)), \quad A^a_0 = 0$$

(4.49)

where $K(\xi)$ and $H(\xi)$ is a function of $\xi = evr$. $K$ and $H$ satisfy

$$\xi^2 {d^2 K \over d\xi^2} = KH^2 + K(K^2 - 1),$$

$$\xi^2 {d^2 H \over d\xi^2} = 2K^2 H + {\lambda \over e^2} H(H^2 - \xi^2)$$

(4.50)

with the boundary conditions

$$K(\xi) \to 1, \quad H(\xi) \to 0 \quad (\xi \to 0)$$

(4.51)

$$K(\xi) \to 0, \quad H(\xi) \to \xi \quad (\xi \to \infty)$$

(4.52)
\[ D_i \phi^a = \frac{\delta_{ai}}{er^2} KH + \frac{r^a r^i}{er^4} (\xi \frac{dH}{d\xi} - H - KH) \tag{4.53} \]

\[ B_i^a = \frac{r^i r^a}{er^4} (1 - K^2 + \xi \frac{dK}{d\xi}) \tag{4.54} \]

magnetic charge

\[ g = \int dS_i B^i = \frac{1}{v} \int dS^a B^{ai} \phi^a = \frac{1}{v} \int d^3x B_i^a D_i \phi^a \]

\[ = \frac{4\pi}{e} \int_0^\infty d\xi \frac{d}{d\xi} \left( \frac{1 - K^2}{\xi} \right) = \frac{4\pi}{e} \tag{4.55} \]

\[ K = 1 + \xi + \cdots \ (\xi \sim 0) \]

**Julia-Zee dyon**

**BPS limit** No analytical solution is not known for (4.50). In the BPS (Bogomol’nyi-Prasad-Sommerfield) limit [6, 7], where \( \lambda \to 0 \), the field equations become simplified. The energy

\[ E = \int d^3x \Theta^{00} \]

\[ = \int d^3x \left\{ \frac{1}{2} (E^{ai} E^{ai} + B^{ai} B^{ai} + (D_j \phi^a)(D_i \phi^a)) \right\} \]

\[ = \frac{1}{2} \int d^3x (E_i^a - D_i \phi^a \sin \alpha)^2 + \frac{1}{2} \int d^3x (B_i^a - D_i \phi^a \cos \alpha)^2 \]

\[ + \sin \alpha \int d^3x E_i^a D_i \phi^a + \cos \alpha \int d^3x B_i^a D_i \phi^a \]

\[ \geq v (\sin \alpha q_e + \cos \alpha q_m) \tag{4.56} \]

where

\[ q_e = \frac{1}{v} \int d^3x \partial_i (E_i^a \phi^a), \quad q_m = \frac{1}{v} \int d^3x \partial_i (B_i^a \phi^a) \tag{4.57} \]

are electric and magnetic charge, respectively. The inequality holds for arbitrary real \( \alpha \). Equality holds for

\[ E_i^a = D_i \phi^a \sin \alpha, \quad B_i^a = D_i \phi^a \cos \alpha \tag{4.58} \]
These eqs. are called the BPS equations. For $\alpha$ with $\tan \alpha = \frac{q_e}{q_m}$, the rhs of (4.56) has extremum with value $v \sqrt{q_e^2 + q_m^2}$. Then we obtain

$$M \geq v \sqrt{q_e^2 + q_m^2} = v|q_e + iq_m|$$  \hspace{1cm} (4.59)

This bound for mass is called the Bogomol’nyi bound.

For the monopole, the BPS equation becomes

$$B^a_i = D_i \phi^a$$  \hspace{1cm} (4.60)

The ansatz (4.48) and (4.49) lead to

$$\xi \frac{dK}{d\xi} = -KH, \quad \xi \frac{dH}{d\xi} = H + 1 - K^2$$  \hspace{1cm} (4.61)

which has the solution

$$K = \frac{\xi}{\sinh \xi}, \quad H = \xi \cosh \xi - 1$$  \hspace{1cm} (4.62)

**Exercise 10.2** Show (4.61).

**Witten effect** The Lagrangian with $\Theta$-term:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a - \frac{\Theta e^2}{32\pi^2} \tilde{F}_{\mu\nu}^a F_{\mu\nu}^a + \frac{1}{2} D^\mu \phi^a D_\mu \phi^a - V(\phi)$$  \hspace{1cm} (4.63)

electromagnetic $U(1)$ subgroup in $G = SO(3)$:

infinitesimal gauge transformation: $U = 1 + in^a T^a$ with $n^a = \frac{\phi^a}{e}$

$$\phi^a T^a \to U \phi^a T^a U^{-1} = \phi^a T^a$$  \hspace{1cm} (4.64)

$$A^a_\mu T^a \to U A^a_\mu T^a U^{-1} + \frac{i}{e} U \partial_\mu U^{-1} = A^a_\mu + \frac{1}{e v} D_\mu \phi^a T^a$$  \hspace{1cm} (4.65)

$$\delta \phi^a = 0, \quad \delta A^a_\mu = \frac{1}{e v} D_\mu \phi^a$$  \hspace{1cm} (4.66)

electric charge number

$$n_e = \int d^3x \left\{ \frac{\partial \mathcal{L}}{\partial (\partial_0 A^a_\mu)} \delta A^a_\mu + \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi^a)} \delta \phi^a \right\}$$

$$= \int d^3x \left\{ -\frac{1}{e v} F^{a0i} D^i \phi^a - \Theta e^2 \tilde{F}^{a0i} D^i \phi^a \right\}$$

$$= \frac{1}{e} q_e - \frac{\Theta e^2}{8\pi^2} q_m$$  \hspace{1cm} (4.67)
Using \( q_m = n_m g \) and \( eg = 4\pi \), we get

\[
q_e = en_e + \frac{e\Theta n_m}{2\pi}
\]  \hspace{1cm} (4.68)

electromagnetic duality in path-integral formalism

GNO duality

Montonen-Olive duality