On the mass-coupling relation of multi-scale quantum integrable models

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Introduction

mass-coupling relation in quantum field theories

the relation between the couplings in Lagrangian and the physical masses.

- UV$\leftrightarrow$ IR
- the chiral limit of QCD: $\Lambda_{QCD} \leftrightarrow$ nucleon mass
- exactly calculable in 2-dim integrable QFT
  $O(3)$ non-linear sigma model [Hasenfratz-Maggiore-Niedermayer 1990]

$$\frac{m}{\Lambda_{\overline{MS}}} = \frac{8}{\mathrm{e}}$$

- put the system in a external magnetic field $h \gg m$
  free energy density $f(h)$
  UV: $f(h) - f(0) = -\frac{h^2}{4\pi}\left(\log \frac{h}{\mathrm{e}^{1/2}\Lambda_{\overline{MS}}} + \cdots\right)$ asymptotically free
  IR: $f(h) - f(0) = -\frac{h^2}{4\pi}\left(\log \frac{h}{m} + 3(\log 2 - \frac{1}{2}) + \cdots\right)$ TBA

- 2-dim sine-Gordon model [A.B. Zamolodchikov 1995]

$$\mathcal{A}_{SG_p} = \frac{1}{2}\int (\partial_a \varphi)^2 d^2x - 2\mu \int \cos\beta\varphi d^2x, \quad \Delta = \frac{\beta^2}{8\pi} = \frac{p}{p+1}$$

- UV: minimal model +relevant perturbation

$$\mathcal{A}_{\mathcal{M}_p} + \lambda \int \Phi_{(1,3)} d^2x$$

- IR: integrable field theory with solitons with mass M
- put the system in a external gauge field coupled with the soliton charge

$$-\lambda = \frac{2\kappa^2(p)}{Q} M^{\frac{4}{p+1}}$$

- affine Toda Field Theories [Fateev 1994]

- There are no results for exact multi-scale mass-coupling relations in the literature.
- The homogenous sine-Gordon (HSG) model are the integrable models with multiple mass scales.
  [FernandezPousa, Gallas, Hollowood, Miramontes, 1996]
  cf. Ising model +mass and magnetic perturbations non-integrable
- we analyze the HSG model both from the UV and IR side, and compare the results to obtain the mass-coupling relation.

UV: perturbed CFT

Homogenegous sine-Gordon model (G/H)

- $SU(n)_k/U(1)^{n-1}$ coset CFT + weight 0 adjoint primary fields ϕ_{adj}

$SU(3)_2/U(1)^2$ generalized parafermions [Gepner]

- central charge $c = \frac{6}{5}$
- adjoint rep of $su(3)$: 8-dims
  weight zero: 2 states $(h, \bar{h}) = (\frac{3}{5}, \frac{3}{5})$
- level-rank duality [Nemeschansky-Yankielowicz]

$$\frac{SU(3)_2}{U(1)^2} = \frac{[SU(2)_1]^3}{SU(2)_3} = \frac{SU(2)_1 \times SU(2)_1}{SU(2)_2} \times \frac{SU(2)_1 \times SU(2)_2}{SU(2)_3}$$

- projected product [Crnkovic-Stanishkov 1989]

$$su(3)_2/u(1)^2 \sim \mathcal{M}_{3,4}[\text{Ising}] \otimes \mathcal{M}_{4,5}[\text{TCI}]$$

$\mathcal{M}_{p,q}$: the minimal model with central charge $c = 1 - \frac{6(p-q)^2}{pq}$.

- $M_{3,4}$ (Ising model): $c = \frac{1}{2}$
  a free real fermion $\psi(z)$, $L^{(1)}(z) = -\frac{1}{2}\psi(z)\partial\psi(z)$
- $M_{4,5}$ (Tricritical Ising model): $c = \frac{7}{10}$
  $N = 1$ superconformal algebra $G(z)$, $L^{(2)}(z)$
- full Virasoro generator $L(z) = L^{(1)}(z) + L^{(2)}(z)$,
  central charge

$$\frac{6}{5} = \frac{1}{2} + \frac{7}{10}$$

- The coset chiral algebra is larger than the Virasoro algebra.
  non-diagonal modular invariant partition function

- There are 4 fields of dimension $(3/5, 3/5)$

$$\Phi_{ij}(z,\bar{z}) = \psi^{(i)}_{-1/2}\bar{\psi}^{(j)}_{-1/2}\Phi(z,\bar{z})\,,$$

$\Phi(z,\bar{z}) \equiv \Phi_{1/10,1/10}(z,\bar{z})$: primary fields in $\mathcal{M}_{4,5}$
$\psi^{(1)}_{-1/2} = \psi_{-1/2}$, $\psi^{(2)}_{-1/2} = \sqrt{5}G_{-1/2}$.

- pCFT description the HSG theory

$$\mathcal{A}^{HSG} = \mathcal{A}^{CFT} - \int d^2x \sum_{i,j=1}^{2} \lambda_i \bar{\lambda}_j \Phi_{ij}(z,\bar{z})\,,$$

- Since the transformations $\lambda_i \to \Lambda\lambda_i$ and $\bar{\lambda}_i \to \Lambda^{-1}\bar{\lambda}_i$ do not change the perturbation we have effectively 3 parameters.

theory on a cylinder with circumference L and length R

$$Z(R,L) = \langle \exp(-\int d^2x \sum_{i,j=1}^{2} \lambda_i \bar{\lambda}_j \Phi_{ij}(z,\bar{z})) \rangle$$

ground state energy $E_0(L)$

$$F(L) = \frac{L}{2\pi} E_0(L) = -\lim_{R\to\infty} \frac{1}{RL} \log Z = -\frac{c}{12} + \sum_{n=1}^{\infty} F_n L^{n(2-2h)}$$

F_n: connected part of the n-point functions of perturbing operators

HSG model: IR decripition

$\frac{SU(3)_2}{U(1)^2}$

- two types of fermionic particles ($a = 1, 2$) with masses m_1 and m_2
  energy $E_a = m_a \cosh\theta$, momenta $p_a = m_a \sinh\theta$ (θ: rapidity)
  light-cone momenta $P_a^{\pm} = m_a e^{\pm\theta}$
- the two particle S-matrices: $\theta = \theta_1 - \theta_2$

$$
\begin{aligned}
S_{12}(\theta) &= \tanh\frac{1}{2}(\theta + \sigma_1 - \sigma_2 - i\frac{\pi}{2}) \\
S_{21}(\theta) &= -\tanh\frac{1}{2}(\theta - \sigma_1 + \sigma_2 - i\frac{\pi}{2}) \\
S_{11}(\theta) &= S_{22}(\theta) = -1
\end{aligned}
$$

σ_1, σ_2: resonance parameters

[Fernandez-Pouza, Gallas, Hollowood, Miramontes 1997]

TBA equation

system in a cylinder with circumference L

pseudo-energy $\epsilon_a(\theta)$ of the a-th particle

TBA equation

$$\epsilon_a(\theta) = m_a L \cosh\theta - K_{ab} * \log(1 + e^{-\epsilon_b})$$

- Kernel: $K_{ab}(\theta) = -i\partial_\theta \log S_{ab}(\theta)$
- convolution $(f * g)(\theta) = \int \frac{d\theta'}{2\pi} f(\theta - \theta') g(\theta')$
- free energy

$$F(L) = -\frac{L}{4\pi^2} \sum_{a=1}^{2} \int_{-\infty}^{\infty} d\theta m_a \cosh\theta \log(1 + e^{-\epsilon_a}) + F_{bulk} L^2$$

$$F_{bulk} = \frac{1}{\pi} m_1 m_2 \cosh(\sigma_1 - \sigma_2)$$

TBA equation (2)

light-cone description of TBA system

- left and right masses

$$\mu_a := \frac{m_a e^{\sigma_a}}{2}, \quad \bar{\mu}_a := \frac{m_a e^{-\sigma_a}}{2}$$

- shifted pseudo-energies $\hat{\epsilon}_a(\theta) = \epsilon(\theta - \sigma_a)$
- TBA eqs with kernel $K(\theta) := \frac{1}{\cosh\theta}$

$$\hat{\epsilon}_1 + K * \log(1 + e^{-\hat{\epsilon}_2}) = L(\bar{\mu}_1 e^{\theta} + \mu_1 e^{-\theta})$$
$$\hat{\epsilon}_2 + K * \log(1 + e^{-\hat{\epsilon}_1}) = L(\bar{\mu}_2 e^{\theta} + \mu_2 e^{-\theta})$$

- free energy

$$F(L) = -\frac{L}{4\pi^2}\sum_{a=1}^{2}\int_{-\infty}^{\infty} d\theta(\bar{\mu}_a e^{\theta} + \mu_a e^{\theta}) \log(1 + e^{-\hat{\epsilon}_a}) + F_{bulk}L^2$$

$F_{bulk} = \frac{1}{2\pi}(\mu_1\bar{\mu}_2 + \mu_2\bar{\mu}_1)$

- symmetry $\mu_1 \leftrightarrow \mu_2$

TBA vs pCFT

- $M_1 \neq 0$, $M_2 = 0$ [Zamolodchikov]
  UV: $M_{4,5} + \lambda_{RSOS}\phi_{1,3}$ $h = \frac{3}{5}$

$$\lambda_{RSOS} = -\kappa^{RSOS} M_1^{2(1-h)}, \quad \kappa_3^{\rm RSOS} = \frac{1}{2(12\pi)^{1/5}} \gamma^{\frac{1}{2}}\left(\frac{2}{5}\right)\gamma^{\frac{1}{2}}\left(\frac{4}{5}\right)$$

  $\gamma(x) := \frac{\Gamma(x)}{\Gamma(1-x)}$

- $M_1 = M_2 = M$ [Fateev]
  UV: $\mathcal{M}_{3,5} + \hat{\lambda}\phi_{1,3}$ $h = \frac{1}{5}$

$$\hat{\lambda} = \hat{\kappa} M^{\frac{8}{5}}, \quad \hat{\lambda}^2 = -\frac{1}{(16\pi)^{2/5}}\left(\frac{5}{16}\right)^2 \gamma\left(\frac{3}{5}\right)\gamma\left(\frac{4}{5}\right)\gamma^{\frac{16}{5}}\left(\frac{1}{4}\right)$$

Mass-coupling relations:Main results

$$\mu_1(\lambda_1,\lambda_2) = B\lambda_1^2(\lambda_1+\sqrt{3}\lambda_2)^{\frac{1}{2}}F(\frac{2\lambda_1}{\lambda_1+\sqrt{3}\lambda_2})$$

$$\mu_2(\lambda_1,\lambda_2) = \frac{B}{4}(\sqrt{3}\lambda_2-\lambda_1)^2(\lambda_1+\sqrt{3}\lambda_2)^{\frac{1}{2}}F(\frac{\sqrt{3}\lambda_2-\lambda_1}{\lambda_1+\sqrt{3}\lambda_2})$$

$$F(z) := {}_2F_1(-\frac{1}{2},\frac{3}{2};3;z)$$

$$B := \kappa\frac{5\pi}{16\cdot 3^{\frac{1}{4}}}$$

$$\kappa := \frac{56(12\pi)^{\frac{1}{4}}}{5^{\frac{3}{2}}}\frac{\Gamma(-\frac{7}{5})\Gamma(\frac{1}{5})}{\Gamma(\frac{12}{5})\Gamma(\frac{4}{5})}$$

- $\lambda_1=0$, $m_2=\kappa(\lambda_2\bar{\lambda}_2)^{5/4}$ [Zamolodchikov]
- $(\lambda_1,\lambda_2)=(\frac{\lambda}{2},\frac{\sqrt{3}\lambda}{2})$ $\mu_1=\mu_2=\frac{B}{\sqrt{2}}F(\frac{1}{2})\lambda^{\frac{5}{2}}$ [Fateev]

We are going to prove the mass coupling relations in the following steps.

- UV-IR operator identification
- Form factor perturbation theory
- UV conserved currents
- Generalized Θ sum rule

UV-IR operator identification:Form Factor approach

- UV: $\Phi_{ij} \Leftrightarrow$ IR: local operators X_{ab}
- IR: the operators are characterized by their form factors:

$$\langle 0|X|\theta_1, \ldots, \theta_n\rangle_{a_1 \ldots a_n} = F^X_{a_1,\ldots,a_n}(\{\theta_i\})\,.$$

[Castro-Alvaredo, Fring, Korff 2000]

$$F^X_{a_1 \ldots a_n}(\{\theta_i\}) = Q^X_{a_1 \ldots a_n}(\{x_i\}) \prod_{j<k} F_{a_j a_k}(\theta_i, \theta_j)\,,$$

- $Q^X_{a_1 \ldots a_n}(\{x_i\})$: polynomials in x_i and $1/x_i$ ($x_i := e^{\theta_i}$)
- the two particle form factors

$$F_{11}(\theta_1,\theta_2) = F_{22}(\theta_1,\theta_2) = -\frac{\sinh\frac{\theta_1-\theta_2}{2}}{2\pi(x_1+x_2)}$$

$$F_{12}(\theta_1,\theta_2) \equiv f(\theta_1-\theta_2), \quad F_{21}(\theta_1,\theta_2) = f(\theta_2-\theta_1)/S_{12}(\theta_2-\theta_1)$$

$f(\theta)$ is the minimal solution of the equation $f(\theta) = S_{12}(\theta) f(\theta+2i\pi)$;

UV-IR operator identification (2)

- the trace of the stress tensor $\Theta = T^{\mu}{}_{\mu}$,

$$Q^{\Theta}_{a_1\ldots a_n}(\{x_i\}) = P(\{x_i\})^2 q_{a_1\ldots a_n}(\{x_i\}) \,,$$

$$P^2 = P^+P^-, \; P^{\pm} = P^{\pm}_{(1)} + P^{\pm}_{(2)} \;, \; P^{\pm}_{(a)} = m_a \sum_{j\in \text{type}\, a} x_j^{\pm 1}$$

- Define four local operators X_{ab} by their form factors:

$$Q^{X_{ab}}_{a_1\ldots a_n} = \hat{P}^{+}_{(a)}\hat{P}^{-}_{(b)} q_{a_1\ldots a_n} \qquad P^{\pm}_{(a)} = m_a \hat{P}^{\pm}_{(a)}$$

- $\Theta = m_1^2 X_{11} + m_2^2 X_{22} + m_1 m_2 (X_{12} + X_{21})$
- free energy $F = \frac{1}{2}\langle\Theta\rangle = \frac{m_1 m_2}{2}\cosh\sigma$ $(\sigma = \sigma_1 - \sigma_2)$

$$\langle X_{11}\rangle = \langle X_{22}\rangle = 0, \quad \langle X_{12} + X_{21}\rangle = \cosh\sigma$$

- operators X_{ab} have conformal weights $(\frac{3}{5}, \frac{3}{5})$

UV vs IR

Θ is expressed in two ways:

$$\Theta = -\frac{4}{5}\sum_{i,j}\lambda_i\bar{\lambda}_j\Phi_{ij} = \sum_{a,b} X_{ab}\,,$$

$\mathcal{F} = -\lim_{V\to\infty}\frac{1}{V}\ln Z = \frac{1}{2}\langle\Theta\rangle$.

$$\begin{aligned}\partial_i\mathcal{F} &= -\langle\Psi_i\rangle\,, \quad \Psi_i = -\bar{\lambda}_j\Phi_{ij}\,,\\ \bar{\partial}_j\mathcal{F} &= -\langle\bar{\Psi}_j\rangle\,, \quad \bar{\Psi}_j = -\lambda_i\Phi_{ij}\,,\end{aligned}$$

the diagonal one particle form factors, $F_{aa}^{X} \equiv F_{aa}^{X}(i\pi, 0)$,

$$\partial_i m_a^2 = -4\pi F_{aa}^{\Psi_i}\,, \quad \bar{\partial}_j m_a^2 = -4\pi F_{aa}^{\bar{\Psi}_j}\,.$$

$$\begin{aligned}\Psi_i &= -X_{11}\,\partial_i\ln m_1 - X_{12}\,\partial_i\ln(m_1 m_2 e^{-\sigma})^{1/2}\\ &\quad -X_{22}\,\partial_i\ln m_2 - X_{21}\,\partial_i\ln(m_1 m_2 e^{\sigma})^{1/2}\,.\end{aligned}$$

UV integrability

the current $\Lambda(z)$ in the chiral algebra: $\bar{\partial}\Lambda(z) = 0$

Its perturbative correction is

$$\bar{\partial}\Lambda(z,\bar{z}) = -\lambda_i\bar{\lambda}_j \oint_z \frac{dw}{2i}\Lambda(z)\Phi_{ij}(w,\bar{z})\,.$$

If $\bar{\partial}\Lambda(z) = \partial(*)$, it gives the integral of motions. [Zamolodchikov]

- energy-momentum tensor $L = L^{(1)} + L^{(2)}$

$$\bar{\partial}L = \pi(1-h)\lambda_i\partial\Psi_i, \quad h = \frac{3}{5}$$

- spin two current

$$J^- = L^{(1)} + \alpha\psi \otimes G(z)\,, \quad \alpha = \frac{\sqrt{5}}{4}\frac{\lambda_1}{\lambda_2}\,,$$

$$\bar{\partial}J^- = v_i\partial\Psi_i\,,\ v_1 = \frac{\pi}{2}\lambda_1,\ v_2 = \frac{\pi}{6}\frac{\lambda_1^2}{\lambda_2}$$

Conservation Laws

- two spin 1 conservation laws $\partial\Psi_i = \bar{\partial}\tau_i$ ($i = 1, 2$)
  τ_i is the linear combinations of L and J^-.
- Lorentz invariance $F^{\Psi_i} \propto P^+$ and $F^{\bar{\Psi}_i} \propto P^-$

$$\partial_i \ln\left(\frac{m_1}{m_2}e^{-\sigma}\right) = \partial_i\left(\frac{\bar{\mu}_1}{\bar{\mu}_2}\right) = 0\,, \quad \bar{\partial}_i \ln\left(\frac{m_1}{m_2}e^{\sigma}\right) = \bar{\partial}_i\left(\frac{\mu_1}{\mu_2}\right) = 0\,.$$

- μ_1/μ_2 depends only on $\eta = \lambda_1/\lambda_2$
  $\bar{\mu}_1/\bar{\mu}_2$ depends only on $\bar{\eta} = \bar{\lambda}_1/\bar{\lambda}_2$.
-

$$[\mathcal{Q}, \bar{J}^-] = \frac{5}{2}v_i\bar{v}_j\partial\Phi_{ij}$$

$$\Phi_{ij} = -\frac{4}{5}(\partial_i \ln m_a)(\bar{\partial}_j \ln m_b)X_{ba}\,.$$

Sum rules

- a spin-2 current $Y^{\mu\nu}$ satisfying $\partial_\mu Y^{\mu\nu} = 0$
- Ψ: a scalar operator with the OPE $\langle Y^{--}(z)\Psi(0)\rangle = \frac{C(0)}{z^2} + \ldots$.

the sum rule

$$\int d^2x \, \langle Y^{+-}(x)\Psi(0)\rangle_c = -\pi C(0)\,,$$

where $\langle\cdot\rangle_c$ stands for the connected part.

Θ- sum rule

- [Cardy]

$$\int d^2x x^2 \langle\Theta(x)\Theta(0)\rangle_c = \frac{4\pi}{3}c$$

- [Delfino-Simonetti 1996]

$$\int d^2x \langle\Theta(x)\Psi(0)\rangle_c = -2\Delta\langle\Psi\rangle$$

Θ-sum rule

$$\int d^2x \langle \Theta(x) \Psi_j(0) \rangle_c = -2h \langle \Psi_j \rangle, \quad h = \frac{3}{5}$$

Applying $\Psi_j = -\bar{\lambda}_i \Phi_{ji}$ and $\Theta = -\frac{4}{5} \lambda_i \bar{\lambda}_j \Phi_{ij}$, we obtain

$$\int d^2x \langle \Psi_i(x) \Psi_j(0) \rangle_c = \frac{3}{2} \langle \Phi_{ij} \rangle$$

free energy Ward identity

$$\partial_i \bar{\partial}_j F = -\frac{5}{2} \langle \Phi_{ij} \rangle$$

$$\mu_a = \frac{\lambda_1^{5/2}}{2} q_a(\eta)$$

J^+-sum rule

OPE

$$J^-(z)\Psi_j(w) = -\frac{M_{ij}\Psi_j(w)}{z-w} + \cdots$$

with $M_{11}=1$, $M_{12}=M_{21}=\frac{1}{2}\eta$ and $M_{22}=0$.

J-sum rule

$$v_i\partial_i\langle\Phi_{kj}\rangle = \int d^2x\langle J^+(x)\Phi_{kj}\rangle_c = \frac{\pi}{2}M_{ki}\langle\Phi_{ij}\rangle\,,$$

the differential equation for q_a:

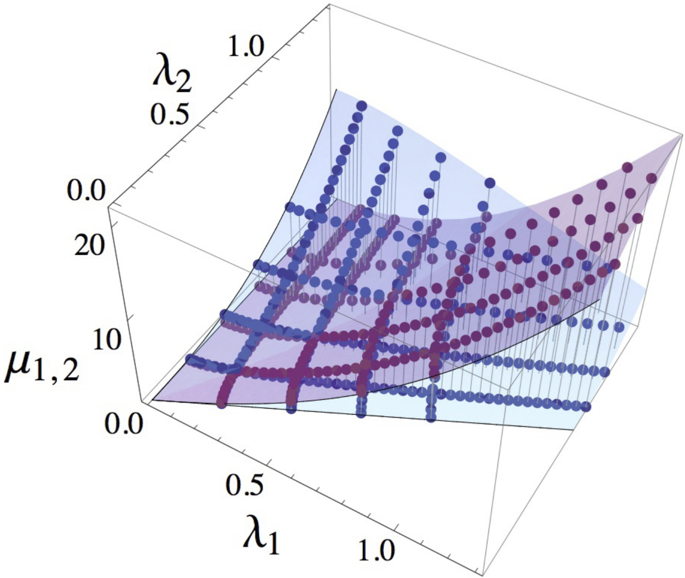
$$\eta^2\left(1-\frac{\eta^2}{3}\right)q_a'' + \eta\left(4-\frac{2\eta^2}{3}\right)q_a' + \frac{5}{4}q_a = 0\,,$$

which is a hypergeometric differential equation.

$\Longrightarrow$

Mass coupling relations

Numerical Check of the mass-coupling relations



Conclusions and Outlook

This is the first result for multi-scale mass-coupling relations.
We developed a new method to calculate the exact mass-coupling relation for multi-scale quantum integrable models.

- form factor perturbation theory
- the construction of conserved tensor currents
- generalization of the Θ sum rule Ward identity of these currents
- an analytic expansion of ten-particle scattering amplitudes of the four-dimensional maximally supersymmetric gauge theory at strong coupling around a $\mathbb{Z}_{10}$-symmetric kinematic point
- HSG models $SU(n)_k/U(1)^{n-1}$