ODE/IM correspondence and modified affine Toda equations

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KI and C. Locke, arXiv:1312.6759, to appear in Nucl. Phys. B

Introduction

- The ODE/IM correspondence is a relation between spectral analysis of ODEs, and the "functional relations" approach to 2d quantum integrable models(IM). [Dorey-Tateo]
- This is an example of the correspondence between classical and quantum integrable models
- has many applications
  - gluon scattering amplitudes in $\mathcal{N}=4$ SYM at strong coupling [Alday-Maldacena]
  - BPS spectrum in $N=2$ SUSY gauge theories [Gaiotto-Moore-Neitzke]
  - gauge/Bethe correspondence [Nekrasov-Shatashvili]

Minimal surface in AdS and quantum integrable system

Alday-Maldacena, Alday-Gaiotto-Maldacena

- String theory in AdS spacetime
- moving frame eq. in AdS=Hitchin system
- linear problem of the Hitchin system $\Longrightarrow$ functional relations for the Stokes coefficients
- Y-system and the TBA system [Alday-Maldacena-Sever-Vieira]
- relation to the Homogeneous Sine-Gordon model $SU(N)_k/U(1)^{N-1}$ [Hatsuda-Ito-Sakai-Satoh]

Hitchin system $\Longrightarrow$ affine Toda field equations (Pohlmeyer reduction)

- AdS_3 : modified sinh-Gordon equations [Alday-Maldacena]
- AdS_4: B_2 affine Toda equations [Burrington-Gao]

- Lukyanov-Zamolodchikov (2010) studied the linear problem associated with the modified sinh-Gordon equation in the context of ODE/IM correspondence ($A_1^{(1)}$)
- The results were generalized to the case of Tzitzéica-Bullough -Dodd equation by Dorey et al. (2012).($A_2^{(2)}$)
- minimal surface in CP_n: A_{n-1} affine Toda equation [Bolton-Woddard]

We will

- Introduce the affine Toda field equation and its linear problem
- Discuss the conformal limit and the Bethe ansatz equations for affine Lie algberas

The general scheme of the ODE/IM correspondence for affine Toda equation is [Dorey-Faldella-Negro-Tateo]

ODE	$\Longleftrightarrow$ ODE/IM	BAE massless TBA	$\Longleftrightarrow$	CFT
$\Uparrow$ Conformal limit		$\Uparrow$ UV limit		
Linear problem	$\Longleftrightarrow$ massive ODE/IM	BAE massive TBA	$\Longleftrightarrow$	massive QFT
$\Uparrow$				
Hitchin system				
$\Uparrow$				
affine Toda equation				

OED/IM correspondence

[Dorey-Tateo, Bazhanov-Lukyanov-Zamolodchikov]

- ODE

$$\left[-\frac{d^2}{dx^2}+\frac{\ell(\ell+1)}{x^2}+x^{2M}-E\right]y(x,E,\ell)=0$$

- large, positive x asymptotics: $y\sim\frac{x^{-\frac{M}{2}}}{\sqrt{2i}}\exp\left(-\frac{x^{M+1}}{M+1}\right)$
- $\{y_k,y_{k+1}\}$ forms a basis of solutions for the ODE: $y_k(x,E,\ell)=\omega^{\frac{k}{2}}y(\omega^{-k}x,\omega^{2k}E,\ell)$ $(\omega=\exp(\frac{2\pi i}{2M+2}))$
- They obeys the Stokes relation

$$C(E,\ell)y_0(x,E,\ell)=y_{-1}(x,E,\ell)+y_1(x,E,\ell)$$

The coeffiecient $C(E,\ell)$ is called the Stokes multiplier.

- small x asymptotics: $\psi(x,E,\ell) \sim x^{\ell+1}$ (other solution is $x^{-\ell}$)
- Take the Wronskian of both sides of the Stokes relation with ψ

$$C(E,\ell) W[y_0,\psi](E,\ell) = W[y_{-1},\psi](E,\ell) + W[y_1,\psi](E,\ell)$$

Setting $D(E,\ell) = W[y_0,\psi]$, the above relation is

$$C(E,\ell) D(E,\ell) = \omega^{-(\ell+\frac{1}{2})} D(\omega^2 E,\ell) + \omega^{\ell+\frac{1}{2}} D(\omega^2 E,\ell)$$

T-Q relation: (D: Q-function (spectral determinant), C: T-function)

- $\psi_+ = \psi(x,E,\ell)$, $\psi_- = \psi(x,E,-\ell-1)$ are linearly independent solutions. The Wronskian $W[\psi_+,\psi_-]$ yields the quantum Wronskian relations for D.
$(2\ell+1) = \omega^{-(\ell+\frac{1}{2})} D_-(\omega^{-1}E) D_+(\omega E) - \omega^{\ell+\frac{1}{2}} D_-(\omega E) D_+(\omega^{-1}E)$
- One can then derive the Bethe ansatz equation from the quantum Wronskian relation.

ODE	I(ntegrable) M(odel)
$\left[-\frac{d^2}{dx^2}+\frac{\ell(\ell+1)}{x^2}+x^{2M}-E\right]y=0$	6-vertex model
energy E	spectral parameter
degree of potential M	anisotropy
angular momentum ℓ	twist paramter
Stokes multiplier $C(E,\ell)$	Transfer matrix (T-function)
spectral determinant $D(E,\ell)$	Q-operator
the Stokes relation	T-Q relation

- relation to (boundary) conformal perturbation theory [BLZ, Bazhanov-Hibberd-Khoroshkin, Kojima]
- generalization to ABCD type [Dorey-Dunning-Masoero-Suzuki-Tateo]

modified sinh-Gordon equation: $A_1^{(1)}$

[Lukyanov-Zamolodchikov 1003.5333]
modified Sinh-Gordon equation

$$\partial_z \partial_{\bar{z}} \eta - e^{2\eta} + p(z)\bar{p}(\bar{z}) e^{-2\eta} = 0, \quad p(z) = z^{2M} - s^{2M}$$

zero curvature condition $[\partial + A, \bar{\partial} + \bar{A}] = 0$

$$\begin{aligned} A &= \frac{1}{2}\partial_z \eta \sigma^3 - e^{\theta}(\sigma^+ e^{\eta} + \sigma^- p e^{-\eta}) \\ \bar{A} &= -\frac{1}{2}\partial_{\bar{z}} \eta \sigma^3 - e^{-\theta}(\sigma^+ e^{\eta} + \sigma^- \bar{p} e^{-\eta}) \end{aligned}$$

asymptotic behavior of $\eta(z, \bar{z})$ at $\rho \to 0, \infty$ $(z = \rho e^{i\phi})$

- $\eta \to M \log \rho$ $(\rho \to \infty)$
- $\eta \to \ell \log \rho$ $(\rho \to 0)$

linear system and its solutions

- linear problem $(\partial + A)\Psi = (\bar{\partial} + \bar{A})\Psi = 0$
- linear problem is invariant under Ω: $\phi \to \phi + \frac{\pi}{M}$, $\theta \to \theta - \frac{i\pi}{M}$
  Π: $\theta \to \theta + i\pi$
- $\rho \to 0$ basis $\Psi_{\pm}(\rho, \phi|\theta)$
- $\rho \to \infty$, from the WKB analysis, subdominant solution is

$$\Xi \sim \begin{pmatrix} e^{\frac{iM\phi}{2}} \\ e^{-\frac{iM\phi}{2}} \end{pmatrix} \exp\left(-\frac{2\rho^{M+1}}{M+1} \cosh(\theta + i(M+1)\phi)\right)$$

- $$\Xi = Q_{-}(\theta)\Psi_{+} + Q_{+}(\theta)\Psi_{-}$$

  $Q_{\pm}(\theta)$ are the Q-function of the quantum Sinh-Gordon model

From MShG to ODE

- take the light-cone limit $\bar{z} \to 0$. Then linear system reduced to a differential equation.

$$\Psi = \begin{pmatrix} e^{\frac{\theta}{2}} e^{\frac{\eta}{2}} \psi \\ e^{\frac{-\eta}{2}} e^{\frac{\theta}{2}} (\partial_z + \partial_z \eta) \psi \end{pmatrix}$$

$$\left[\partial_z^2 - u - e^{\theta} p\right] \psi = 0, \quad u = (\partial_z \eta)^2 - \partial_z^2 \eta$$

- conformal limit: $z \to 0$, $\theta \to \infty$

$$x = z e^{\frac{\theta}{M+1}}, \quad E = s^{2M} e^{\frac{2\theta M}{1+M}}, \quad \text{fixed}$$

$$\left[-\partial_x^2 + \frac{\ell(\ell+1)}{x^2} + x^{2M}\right] \psi = E\psi$$

Schrödinger type ODE: [Dorey-Tateo, Bazhanov-Lukyanov-Zamolodchikov]

affine Toda field equations (1)

$\mathfrak{g}$: a simple Lie algebra of rank r

$$
\begin{aligned}
[E_\alpha, E_\beta] &= N_{\alpha,\beta} E_{\alpha+\beta}, \quad \text{for } \alpha+\beta \neq 0, \\
[E_\alpha, E_{-\alpha}] &= \frac{2\alpha \cdot H}{\alpha^2}, \\
[H^i, E_\alpha] &= \alpha^i E_\alpha.
\end{aligned}
$$

$\alpha_1, \cdots, \alpha_r$: the simple roots of $\mathfrak{g}$
$\alpha_1^\vee, \cdots, \alpha_r^\vee$: simple coroots
$\alpha_0 = -\theta$ (θ: the highest root)
(dual) Coxter labels: $\sum_{i=0}^r n_i \alpha_i = \sum_{i=0}^r n_i^\vee \alpha_i^\vee = 0$.
(dual) Coxter number$h$, $h^\vee$:

$$
h = \sum_{i=0}^r n_i, \quad h^\vee = \sum_{i=0}^r n_i^\vee.
$$

affine Toda field equations (2)

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi \cdot \partial_\mu \phi - \left(\frac{m}{\beta}\right)^2 \sum_{i=0}^{r} n_i \left[\exp(\beta\alpha_i \cdot \phi) - 1\right],$$

$$\partial^\mu \partial_\mu \phi + \left(\frac{m^2}{\beta}\right) \sum_{i=0}^{r} n_i \alpha_i \exp(\beta\alpha_i \phi) = 0\,.$$

complex coordinates: $z = \frac{1}{2}(x^0 + ix^1), \quad \bar{z} = \frac{1}{2}(x^0 - ix^1)$
conformal transformation ($\rho^\vee$: co-Weyl vector)

$$z \to \tilde{z} = f(z)\,, \quad \phi \to \tilde{\phi} = \phi - \frac{1}{\beta}\rho^\vee \log(\partial f \bar{\partial}\bar{f})\,,$$

modified affine Toda equations:

$$\partial\bar{\partial}\phi + \left(\frac{m^2}{\beta}\right)\left[\sum_{i=1}^{r} n_i \alpha_i \exp(\beta\alpha_i\phi) + p(z)\bar{p}(\bar{z}) n_0 \alpha_0 \exp(\beta\alpha_0\phi)\right] = 0,$$

$$p(z) = (\partial f)^h, \quad \bar{p}(\bar{z}) = (\bar{\partial}\bar{f})^h.$$

Lax formalism

- The modified affine Toda equation can be expressed as a linear problem: $(\partial + A)\Psi = 0$ and $(\bar{\partial} + \bar{A})\Psi = 0$.

$$A = \frac{\beta}{2}\partial\phi \cdot H + me^{\lambda}\left\{\sum_{i=1}^{r}\sqrt{n_i^{\vee}}E_{\alpha_i}e^{\frac{\beta}{2}\alpha_i\phi} + p(z)\sqrt{n_0^{\vee}}E_{\alpha_0}e^{\frac{\beta}{2}\alpha_0\phi}\right\},$$

$$\bar{A} = -\frac{\beta}{2}\bar{\partial}\phi \cdot H - me^{-\lambda}\left\{\sum_{i=1}^{r}\sqrt{n_i^{\vee}}E_{-\alpha_i}e^{\frac{\beta}{2}\alpha_i\phi} + \bar{p}(\bar{z})\sqrt{n_0^{\vee}}E_{-\alpha_0}e^{\frac{\beta}{2}\alpha_0\phi}\right\}$$

- zero-curvature condition: $[\partial + A, \bar{\partial} + \bar{A}] = 0 \Longrightarrow$ affine Toda field equations

symmetries and $p(z)$

- Motivated by the ODE/IM correspondence, we put

$$p(z) = z^{hM} - s^{hM}, \quad \bar{p}(\bar{z}) = \bar{z}^{hN} - \bar{s}^{hM}$$

- h: the Coxeter number, and M is some positive real parameter
- We define the transformation $\hat{\Omega}_k$

$$z \to z e^{\frac{2\pi k i}{hM}}$$

$$s \to s e^{\frac{2\pi k i}{hM}}$$

$$\lambda \to \lambda - \frac{2\pi k i}{hM}$$

- The quation of motion and linear problem are invarinant under $\hat{\Omega}_k$ for integer k.

asymptotic behavior of the Toda field

- In the large $|z|$ limit, we assume that the asymptotic solution to the modified affine Toda equation is

$$\phi(z,\bar{z}) = \frac{M}{\beta}\rho^\vee \log(z\bar{z}) + O(1)$$

- For small $|z|$, we assume logarithmic behavior, with expansion

$$\phi(z,\bar{z}) = g\log(z\bar{z}) + \phi^{(0)}(g) + \gamma(z,\bar{z},g) + \sum_{i=0}^{r} \frac{C_i(g)}{(c_i(g)+1)^2}(\bar{z}\bar{z})^{c_i(g)+1} + \cdots .$$

- Substituting this expansion into the Toda equation, we can determine the constants C_i
- The exponents are found to be $c_i + 1 = 1 + \beta\alpha_i \cdot g > 0$.

$A_r^{(1)}$ modified affine Toda [KI-Locke, Adamopuolou-Dunning]

- This is the simplest algebra to start with, and includes the sinh-Gordon model as a specific example
- the fundamental representation with highest weight ω_1 weights are $h_1 = \omega_1$, $h_i = \omega_i - \omega_{i+1}$, $h_{r+1} = -\omega_r$, where ω_i are the fundamental weights defined by $\omega_i \cdot \alpha_j^\vee = \delta_{ij}$
- The linear problem $(\partial_z + A)\Psi = 0$, $\Psi = {}^t(\psi_1, \cdots, \psi_{r+1})$

holomorphic connection:

$$A = \begin{pmatrix} \frac{\beta}{2}h_1 \cdot \partial\phi & me^\lambda e^{\frac{\beta}{2}\alpha_1\cdot\phi} & 0 & \cdots & 0 \\ 0 & \frac{\beta}{2}h_2 \cdot \partial\phi & me^\lambda e^{\frac{\beta}{2}\alpha_2\cdot\phi} & & \vdots \\ & & \ddots & & \\ \vdots & & & \frac{\beta}{2}h_r \cdot \partial\phi & me^\lambda e^{\frac{\beta}{2}\alpha_r\cdot\phi} \\ me^\lambda p(z) e^{\frac{\beta}{2}\alpha_0\cdot\phi} & \cdots & & 0 & \frac{\beta}{2}h_{r+1} \cdot \partial\phi \end{pmatrix}.$$

- gauge transformation: $U = diag(e^{-\frac{\beta}{2}h_1\cdot\phi}, \cdots, e^{-\frac{\beta}{2}h_{r+1}\cdot\phi})$

$$\tilde{A} = UAU^{-1} + U\partial U^{-1}, \quad \tilde{\Psi} = U\Psi,$$

$$\tilde{A} = \begin{pmatrix} \beta h_1\partial\phi & me^{\lambda} & 0 & \cdots & 0 \\ 0 & \beta h_2\partial\phi & me^{\lambda} & & \vdots \\ & & \ddots & & \\ \vdots & & & \beta h_r\partial\phi & me^{\lambda} \\ me^{\lambda}p(z) & & & 0 & \beta h_{r+1}\partial\phi \end{pmatrix}.$$

- the linear problem becomes a single $(r+1)$-th order differential equation

$$D(h_{r+1})\cdots D(h_1)\tilde{\psi}_1 = (-me^{\lambda})^h p(z)\tilde{\psi}_1.$$

$D(h) \equiv \partial + \beta h \cdot \partial\phi$

- scalar Lax operator (Drinfeld-Sokolov reduction)

- For the barred linear equation, a different gauge transformation is used to simplify the equations

$$U = diag(e^{\frac{\beta}{2}h_1\cdot\phi}, \cdots, e^{\frac{\beta}{2}h_{r+1}\cdot\phi}), \quad \tilde{\tilde{\Psi}} = U\Psi$$

- The full lienar problem gives the differential equations

$$D(h_{r+1})\cdots D(h_1)\psi = (-me^{\lambda})^h p(z)\psi$$
$$\bar{D}(-h_1)\cdots \bar{D}(-h_{r+1})\bar{\psi} = (me^{-\lambda})^h \bar{p}(\bar{z})\bar{\psi}$$

where $\psi = \tilde{\psi}_1$ and $\bar{\psi} = \tilde{\tilde{\psi}}_{r+1}$

the linear equation asymptotics

- For small $|z|$, using the asymptotic behavior of ϕ and substituting $\psi \sim z^{\mu}$, the indicial equation gives

$$\mu_i = i - \beta h_{i+1} \cdot g \quad \text{for} \quad i = 0, 1, \ldots, r\,.$$

- orderd $\mu_i < \mu_{i+1}$
- For large $|z|$ limit, the relevant differential eq. becomes

$$(\partial^{r+1} + (-1)^r (me^{\lambda})^h p(z))\psi = 0,$$

- a WKB analysis gives the unique asymptotically decaying solution in the Stokes sector $|\arg z| < \frac{(r+2)\pi}{(r+1)(M+1)}$

$$\psi \sim z^{-\frac{rM}{2}} \exp\left(-\frac{z^{M+1}}{M+1} me^{\lambda} + g(\bar{z})\right),$$

Massive ODE/IM correspondence

- For small $|z|$ solution $\psi^{(i)} \sim z^{\mu_i}$ define the vector $\Psi^{(i)}$ with

$$(\Psi^{(i)})_j \sim \delta_{ij} (\bar{z}/z)^{\frac{\beta}{2} h_i \cdot g}.$$

- For large $|z|$ the solution is

$$\Xi(\rho, \theta|\lambda) \sim C \begin{pmatrix} e^{-\frac{irM\theta}{4}} \\ e^{-\frac{i(r-2)M\theta}{4}} \\ \vdots \\ e^{\frac{irM\theta}{4}} \end{pmatrix} \exp\left(-\frac{2\rho^{M+1}}{M+1} \, m \cosh(\lambda + i\theta(M+1)) \right)$$

- we can expand Ξ as

$$\Xi = \sum_{i=0}^{r} Q_i(\lambda) \Psi^{(i)}.$$

For $A_1^{(1)}$ (sinh-Gordon) $A_2^{(2)}$ (Tzitzéica-Bullough-Dodd), the Q-coefficients correspond to the Q-function of a 2D massive QFT.
$A_r^{(1)}$: KI-Locke, Adamopoulou-Dunning, 1401.1187

Conformal Limit and ODE/IM correspondence

- First we take the light-cone limit $\bar{z} \to 0$ and we define the conformal limit $z \to 0$, $\lambda \to \infty$ with fixed

$$x = (me^{\lambda})^{1/(M+1)} z, \quad E = s^{hM} (me^{\lambda})^{hM/(M+1)}$$

- The differential equation becomes

$$\left[D_x(h_{r+1}) \cdots D_x(h_1) - (-1)^h p(x, E) \right] \psi(x, E, g) = 0$$

where $D_x(a) = \partial_x + \beta \frac{a \cdot g}{x}$ and $p(x, E) \equiv x^{hM} - E$.

- This is the ODE for A_r-type Lie algebra Suzuki, Dorey-Dunning-Tateo
- By writing out the unique asymptotically decaying solution $\xi(x, E, g)$ to this equation in terms of the small x basis $\chi^{(i)} \sim x^{\mu_i} + \mathcal{O}(x^{\mu_i + h})$, we have $\xi(x, E, g) = \sum_{i=0}^{r} Q^{(i)}(E) \chi^{(i)}(x, E, g)$

- Symanzik rotation $\psi_k(x,E,g)=\psi(\omega^k x,\Omega^k E,g)$ with $\Omega=\exp(i\frac{2\pi M}{M+1})$ and $\omega=\exp(i\frac{2\pi}{h})$
- auxiliary functions: $\psi^{(a)}=W^{(a)}(\psi_{\frac{1-a}{2}},\cdots,\psi_{\frac{a-1}{2}})$ $(a=2,\cdots,r)$
- A_n ψ-system (Plücker relations)

$$\psi^{(a-1)}\psi^{(a+1)}=W[\psi^{(a)}_{-\frac{1}{2}},\psi^{(a)}_{\frac{1}{2}}],\quad \psi^{(0)}=\psi^{(n)}=1$$

- quantum Wronskian relation

$$Q^{(a+1)}Q^{(a-1)}=\omega^{\frac{1}{2}(\mu_a-\mu_{a-1})}Q^{(a)}_{-\frac{1}{2}}\bar{Q}^{(a)}_{\frac{1}{2}}-\omega^{\frac{1}{2}(\mu_{a-1}-\mu_a)}Q^{(a)}_{\frac{1}{2}}\bar{Q}^{(a)}_{-\frac{1}{2}}$$

- Bethe ansatz equation

$$\omega^{\mu_{i-1}-\mu_i}\frac{Q^{(i-1)}_{-1/2}(E^{(i)}_n)Q^{(i)}_{1}(E^{(i)}_n)Q^{(i+1)}_{-1/2}(E^{(i)}_n)}{Q^{(i-1)}_{1/2}(E^{(i)}_n)Q^{(i)}_{-1}(E^{(i)}_n)Q^{(i+1)}_{1/2}(E^{(i)}_n)}=-1\,.$$

where $E^{(i)}_n$ are zeros of $Q^{(i)}(E)$.

Other affine Lie algberas[KI-Locke]

- We will consider the other affine Lie algebras and find the (pseudo-)diffrential equations associated to the linear problem for the fundamental representation.

$A_r^{(1)}$	$D(\mathbf{h})\psi = (-me^{\lambda})^h p(z)\psi$
$D_r^{(1)}$	$D(\mathbf{h}^\dagger)\partial^{-1}D(\mathbf{h})\psi = 2^{r-1}(me^{\lambda})^h\sqrt{p(z)}\partial\sqrt{p(z)}\psi$
$B_r^{(1)}$	$D(\mathbf{h}^\dagger)\partial D(\mathbf{h})\psi = 2^{r}(me^{\lambda})^h\sqrt{p(z)}\partial\sqrt{p(z)}\psi$
$A_{2r-1}^{(2)}$	$D(\mathbf{h}^\dagger)D(\mathbf{h})\psi = -2^{r-1}(me^{\lambda})^h\sqrt{p(z)}\partial\sqrt{p(z)}\psi$
$C_r^{(1)}$	$D(\mathbf{h}^\dagger)D(\mathbf{h})\psi = (me^{\lambda})^h p(z)\psi$
$D_{r+1}^{(2)}$	$D(\mathbf{h}^\dagger)\partial D(\mathbf{h})\psi = 2^{r+1}(me^{\lambda})^{2h} p(z)\partial^{-1}p(z)\psi$
$A_{2r}^{(2)}$	$D(\mathbf{h}^\dagger)\partial D(\mathbf{h})\psi = -2^{r}\sqrt{2}(me^{\lambda})^h p(z)\psi$
$G_2^{(1)}$	$D(\mathbf{h}^\dagger)\partial D(\mathbf{h})\psi = 8(me^{\lambda})^h\sqrt{p(z)}\partial\sqrt{p(z)}\psi$
$D_4^{(3)}$	$D(\mathbf{h}^\dagger)\partial D(\mathbf{h})\psi + (\omega+1)2\sqrt{3}(me^{\lambda})^4 D(\mathbf{h}^\dagger)p(z)$ $-(\omega+1)2\sqrt{3}(me^{\lambda})^4 pD(\mathbf{h}) - 8\sqrt{3}\omega(me^{\lambda})^3 D(-h_1)\sqrt{p}\partial\sqrt{p}D(h_1)$ $+(\omega-1)^3 12(me^{\lambda})^8 p\partial^{-1}p\}\psi = 0$

$D(\mathbf{h}) = D(h_r)\cdots D(h_1)$, $D(\mathbf{h}^\dagger) = D(-h_1)\cdots D(-h_r)$ for $\mathbf{h} = (h_r,\cdots,h_1)$

Langlands duality

- In Dorey-Dunning-Masoero-Suzuki-Tateo (2007), they found a set of pseudo-differential equations associated to classical Lie algebras

affine Toda equation	ODE(Dorey et al.)
$A_r^{(1)}$	A_r
$(B_r^{(1)})^\vee = A_{2r-1}^{(2)}$	B_r
$(C_r^{(1)})^\vee = D_{r+1}^{(2)}$	C_r
$D_r^{(1)}$	D_r

- modified affine Toda equation for the Langlands dual $(\hat{\mathfrak{g}})^\vee$ corresponds to the $\mathfrak{g}$-type Bethe ansatz equation

Lie algebra isomorphism

To check our formalism, it is worthwhile to verify some isomorphisms.

- $D_2 = A_1 \oplus A_1$

  $$A_{D_2} = 1_2 \otimes A'_{A_1} + A_{A_1} \otimes 1_2 \Leftrightarrow A_{A_1 \oplus A_1} = A_{A_1} \oplus A'_{A_1}$$

  which is equivalent to $\psi_{D_2} = \psi\psi'$.
  - ODE for A_1: $D(-h_1)D(h_1)\psi = me^{\lambda}p(z)\psi$
    ODE for A'_1: $D(-h_2)D(h_2)\psi' = me^{\lambda}p(z)\psi'$
  - ODE for D_2:
    $D(-h_1)D(-h_2)\partial^{-1}D(h_2)D(h_1)\psi_{D_2} = 4(me^{\lambda})^2\sqrt{p}\partial\sqrt{p}\psi_{D_2}$
- $D_3 = A_3$ spin rep. of D_3=vector rep, of A_3
- $B_2 = C_2$

Outlook

- Investigate effects of using $p(z) = (z^{hM/K} - s^{hM/K})^K$.
- ODE/IM for affine Lie algebra of type $B_r^{(1)}$, $C_r^{(1)}$
- $A_{2r}^{(2)}$
  $r = 1$ Tzitzéica-Bullough-Dodd model [Dorey-Faldella-Negro-Tateo]
- exceptional type ODE/IM($G_2^{(1)}$ and $D_4^{(3)}$, $F_4^{(1)}$ and $E_6^{(2)}$, $E_r^{(1)}$)
- Matrix ODE/IM correspondence [Sun] auxiliary function ψ_a correspond to the highest weight component in the rep with h.w. ω_a.
  ψ-system $\Longrightarrow$ quantum Wronskian $\Longrightarrow$ BAE
- super affine Toda field theory based on affine Lie superalgebra
- generalized Drinfeld-Sokolov reduction
  [Balog-Feher-O'Raifeartaigh-Forgacs-Wipf]
- minimal surface in CP^n and other applications (vortex, Nekrasov-Shatashvili etc.)

ψ-system for G_2

- $D_4^{(3)} = (G_2^{(1)})^\vee$

$$\iota(\Psi^{(1)}_{1/2} \wedge \Psi^{(1)}_{1/2}) = \Psi^{(2)}$$
$$\iota(\Psi^{(2)}_{1/6} \wedge \Psi^{(2)}_{1/6}) = \Psi^{(1)}_{-1/3} \otimes \Psi^{(1)}_{0} \otimes \Psi^{(1)}_{1/3}$$

[Dorey-Dunning-Masoero-Suzuki-Tateo]

- $G_2^{(1)}$

$$\iota(\Psi^{(2)}_{1/2} \wedge \Psi^{(2)}_{1/2}) = \Psi^{(1)}$$
$$\iota(\Psi^{(1)}_{1/2} \wedge \Psi^{(1)}_{1/2}) = \Psi^{(2)} \otimes \Psi^{(2)} \otimes \Psi^{(2)}$$