Homework (2) of the intensive lecture

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4 Reconstructing the deformed prepotential

Let us treat the modified Mathieu equation:

$$\left(-\hbar^2 \frac{d^2}{dx^2} + 2\Lambda^2 \cosh x\right)\psi(x) = E\psi(x).$$

The quantum period integrals are given by

$$A_{2k}(E) = \oint_{C(a,b)} P_{2k}(x)dx, \qquad A_{2k}^{\mathrm{D}}(E) = \frac{1}{2\pi i}\int_{-\pi i}^{\pi i} P_{2k}(x)dx.$$

As I explain in the lecture, the dual quantum period $A_{2k}^{\mathrm{D}}(E)$ is computed easily in the large E or the small Λ limit. This fact allows us to fix the deformed prepotential from the dual quantum period as follows.

(1) The Riccati equation for the WKB solution is given by

$$P(x)^2 - i\hbar P'(x) = Q(x).$$

For $Q(x) = E - 2\Lambda^2 \cosh x$, we can solve this equation order by order in Λ. We assume the perturbative expansion

$$P(x) = \sum_{n=0}^{\infty} \Lambda^{2n} Y_n(x).$$

At the zero-th order ($\Lambda = 0$), we find two solutions $Y_0(x) = \pm\sqrt{E}$ because it corresponds to the free particle. Set $Y_0(x) = \sqrt{E}$, then the n-th order correction satisfies

$$Y_n'(x) + ikY_n(x) = R_n(x), \qquad k := \frac{2\sqrt{E}}{\hbar},$$

where $R_n(x)$ is determined by the lower order corrections. Confirm that

$$Y_n(x) = e^{-ikx}\int^x e^{ikx'} R_n(x')dx'$$

solves the above inhomogeneous differential equation.

(2) For $n = 1$, we have

$$R_1(x) = \frac{2}{i\hbar}\cosh x.$$

Confirm that the special solution to this equation is given by

$$Y_1(x) = -\frac{4\sqrt{E}\cosh x + 2i\hbar\sinh x}{4E + \hbar^2},$$

where we determined an integration constant so that the special solution does not include the homogeneous solution e^{-ikx}.

(3) For $n \geq 2$, we have

$$R_n(x) = \frac{1}{i\hbar}\sum_{m=1}^{n-1} Y_m(x)Y_{n-m}(x).$$

Confirm that

$$Y_2(x) = -\frac{4E^2 + 5E\hbar + \hbar^2 + E(4E - 5\hbar^2)\cosh 2x + i\sqrt{E}\hbar(8E - \hbar^2)\sinh 2x}{\sqrt{E}(E + \hbar^2)(4E + \hbar^2)^2}$$

is a solution to the second order equation.

(4) Pushing this computation, we can find $Y_n(x)$ systematically. Let us define

$$A^{\mathrm{D}}(E;\hbar) := \sum_{k=0}^{\infty} \hbar^{2k} A^{\mathrm{D}}_{2k}(E).$$

This is formally written as

$$A^{\mathrm{D}}(E;\hbar) = \frac{1}{2\pi i}\int_{-\pi i}^{\pi i} P(x)dx = \sum_{n=0}^{\infty} \Lambda^{2n}\frac{1}{2\pi i}\int_{-\pi i}^{\pi i} Y_n(x)dx.$$

Evaluating these integrals, we obtain

$$A^{\mathrm{D}}(E;\hbar) = \sqrt{E} - \frac{1}{\sqrt{E}(4E + \hbar^2)}\Lambda^4 - \frac{60E^2 + 35E\hbar^2 + 2\hbar^4}{4E^{3/2}(E + \hbar^2)(4E + \hbar^2)^3}\Lambda^8 + O(\Lambda^{12}).$$

Now the quantum corrections are resummed! By solving this equation with respect to E, we can express E in terms of $a = A^{\mathrm{D}}(E;\hbar)$. Moreover, by using

$$E = -\frac{\Lambda}{4}\frac{\partial F(a;\hbar)}{\partial \Lambda},$$

we can finally reconstruct the deformed prepotential as

$$F(a;\hbar) = f(a;\hbar) - 4a^2\log\Lambda - \frac{2}{4a^2 + \hbar^2}\Lambda^4 - \frac{20a^2 - 7\hbar^2}{4(a^2 + \hbar^2)(4a^2 + \hbar^2)^3}\Lambda^8 + O(\Lambda^{12}),$$

where $f(a;\hbar)$ is a function depending on a and $\hbar$ but not on Λ. This part corresponds to the "perturbative part" in the Nekrasov partition function. Confirm the above computation by yourself.

5 A toy model for quasi-normal mode problems

Let us consider the following wave equation:

$$\left(\frac{d^2}{dx^2} + \omega^2 - \frac{\ell(\ell+1)}{2\cosh^2 x}\right)\psi(x) = 0 \qquad (\ell > 0).$$

If the QNM boundary condition

$$\psi(x) \sim \begin{cases} e^{i\omega x} & (x \to \infty) \\ e^{-i\omega x} & (x \to -\infty) \end{cases}$$

is imposed, then only the discrete complex values of ω are allowed. In this toy model, the exact QNM eigenvalues are known:

$$\omega_n = \pm\sqrt{\frac{\ell(\ell+1)}{2} - \frac{1}{4}} - i\left(n + \frac{1}{2}\right), \qquad n = 0, 1, 2, \dots$$

See §4.1 in [Berti et al., arXiv: 0905.2975] for instance.

Here we treat this problem by the WKB method. We define

$$\hbar = \left(\frac{2}{\ell(\ell+1)}\right)^{1/2}, \qquad E = (\hbar\omega)^2.$$

The wave equation now takes the form of the Schrödinger equation

$$\left(\hbar^2\frac{d^2}{dx^2} + E - \frac{1}{\cosh^2 x}\right)\psi(x) = 0.$$

(1) Assume $0 < E < 1$. We want to compute the integral

$$\widetilde{A}_0(E) := 2\int_a^b \sqrt{\frac{1}{\cosh^2 x} - E}\, dx = \frac{1}{i}\oint_{C(a,b)} P_0(x)dx,$$

where a and b are two real turning points with $b = -a > 0$. By changing the integration variable, show

$$\widetilde{A}_0(E) = 2\pi(1 - \sqrt{E}).$$

This result is analytically continued to the complex domain for E. Then, the Bohr-Sommerfeld quantization condition

$$\widetilde{A}_0(E) = 2\pi i\hbar\left(n + \frac{1}{2}\right), \qquad n = 0, \pm 1, \pm 2, \dots$$

gives the approximate QNM eigenvalues

$$E_n^{\mathrm{BS}} = \left[1 - i\hbar\left(n + \frac{1}{2}\right)\right]^2$$

(2) The second order corrections are given by

$$\widetilde{A}_2(E) = \frac{1}{i}\oint_{C(a,b)} P_2(x)dx.$$

Show

$$\widetilde{A}_2(E) = \left(\frac{E^2}{6}\frac{d^2}{dE^2} + \frac{E}{3}\frac{d}{dE} - \frac{1}{8}\right)\widetilde{A}_0(E),$$

and solve the quantization condition

$$\widetilde{A}_0(E) + \hbar^2\widetilde{A}_2(E) = 2\pi i\hbar\left(n + \frac{1}{2}\right), \qquad n = 0, \pm 1, \pm 2, \ldots.$$

(3) Compare the above second order result with the exact result:

$$E_n = (\hbar\omega_n)^2 = \left[\pm\sqrt{1 - \frac{\hbar^2}{4}} - i\hbar\left(n + \frac{1}{2}\right)\right]^2, \qquad n = 0, 1, 2, \ldots.$$