Homework (1) of the intensive lecture

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$\boxed{1}$ Let us consider the "zero-dimensional Euclidean path integral" of ϕ^4-theory:

$$Z(g) = \int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\phi^2 - \frac{g}{4}\phi^4\right) \qquad (g > 0). \tag{1}$$

We can evaluate this integral by perturbation theory.

(1) By expanding the integrand around $g = 0$, show that the all-order perturbative expansion of (1) is given by

$$Z^{\mathrm{pert}}(g) = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(2k + \frac{1}{2})}{\sqrt{\pi}k!} g^k. \tag{2}$$

(2) Show that the radius of convergence of the perturbative series (2) is zero.

(3) Compute the Borel sum $Z^{\mathrm{B}}(g)$ of the perturbative series (2).

(4) Confirm $Z^{\mathrm{B}}(g) = Z(g)$ numerically by choosing some values of g as you want.

$\boxed{2}$ If the coupling constant g in the path integral (1) is negative, we encounter the singularity problem of the Borel resummation. This problem is related to the fact that the path integral (1) is no longer convergent for $g = -\lambda < 0$. As in the lecture, we define two modified Borel sums $Z_{\pm}^{\mathrm{B}}(g = -\lambda)$ by deforming the integration contour.

Show that the difference $Z_{+}^{\mathrm{B}}(g = -\lambda) - Z_{-}^{\mathrm{B}}(g = -\lambda)$ behaves non-perturbatively as

$$Z_{+}^{\mathrm{B}}(g = -\lambda) - Z_{-}^{\mathrm{B}}(g = -\lambda) \sim i\sqrt{2} \exp\left(-\frac{1}{4\lambda}\right) \qquad (0 < \lambda \ll 1).$$

If it is hard to show it analytically, confirm it numerically.

$\boxed{3}$ Consider the quartic oscillator:

$$\left(-\hbar^2 \frac{d^2}{dx^2} + x^2 + gx^4\right)\psi(x) = E\psi(x) \qquad (g > 0).$$

(1) The leading contribution to the Bohr-Sommerfeld quantization condition is

$$A_0(E) = 2\int_{-b}^{b} \sqrt{E - x^2 - gx^4}\,dx = 4\int_{0}^{b} \sqrt{E - x^2 - gx^4}\,dx,$$

where the turning point b is given by

$$b = \left(\frac{\sqrt{4gE+1} - 1}{2g} \right)^{1/2}.$$

Changing the integration variable by $\xi = x^2$, we rewrite it as

$$A_0(E) = 2\int_0^{\alpha} \sqrt{\frac{E - \xi - g\xi^2}{\xi}}\,d\xi,$$

where $\alpha = b^2$. Compute $A_0(E)$ analytically. You can use an integration formula

$$\int_{a_2}^{a_3} \sqrt{\frac{(z-a_1)(a_3-z)}{z-a_2}}\,dz = \frac{2\sqrt{a_3-a_1}}{3}\Bigg[(a_2-a_1)\mathbb{K}\left(\frac{a_3-a_2}{a_3-a_1}\right)$$
$$+(a_1+a_3-2a_2)\mathbb{E}\left(\frac{a_3-a_2}{a_3-a_1}\right)\Bigg] \qquad (a_1 < a_2 < a_3),$$

where $\mathbb{K}(m)$ and $\mathbb{E}(m)$ are the complete elliptic integrals.

(2) Check if the above result reproduces the harmonic oscillator in the limit $g \to 0$.

(3) Show that $A_0(E)$ satisfies the following Picard-Fuchs equation:

$$\left[4E(1+4gE)\frac{d^2}{dE^2} + 3g\right]A_0(E) = 0.$$

(4) Show that the second order quantum correction $A_2(E)$ is given by

$$A_2(E) = -\left[\frac{g}{2}\frac{d}{dE} + \left(\frac{1}{6} + 2gE\right)\frac{d^2}{dE^2}\right]A_0(E).$$