Scattering · Resonance · Bound states on lattice

— including review —

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Strangeness workshop @ KKR Hotel Atami
February 27th, 2009

1. Introduction

Motivation :

Understand hadron interactions from (lattice) QCD

We need to study not only scattering state, but resonance (bound) state on lattice, simultaneously.

Review

Resent results of a_0 and $\delta(p)$ from Lüscher's finite volume method

Proposed methods for resonance (bound) state

Outline

1. ~~Introduction~~
2. Finite volume method
3. Scattering state
4. Resonance state
5. Bound state
6. More complex scattering state
7. Summary

2. Finite volume method

Lüscher, CMP105 153(1986)
NPB354 531(1991)

Finite volume L^3 in center of mass system
with periodic boundary condition in spatial directions

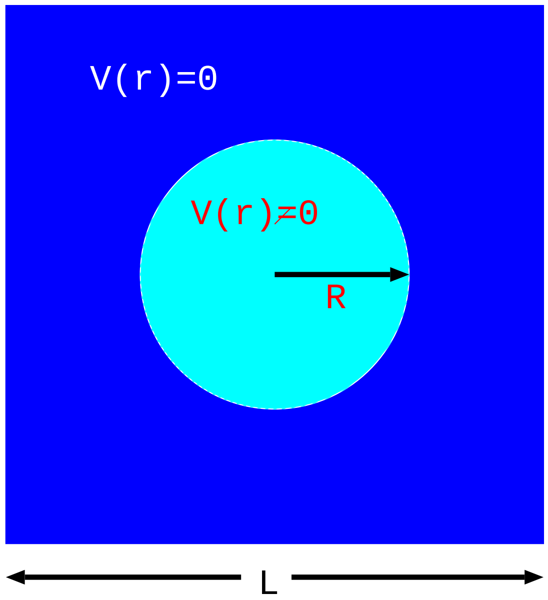
$$\mathbf{p} = 2\pi/L \cdot \mathbf{n}$$

Important assumption

1. Two-pion interaction is localized.
   $\longrightarrow$ Interaction range R exists.

$$V_p(r) \begin{cases} \neq 0 & (r \leq R) \\ = 0 \ (\sim e^{-cr}) & (r > R) \end{cases}$$

2. $V_p(r)$ is not affected by boundary. $\rightarrow$ $R < L/2$



Two-particle wave function $\phi(\mathbf{r})$ satisfies Helmholtz equation

$$\left(\nabla^2 + p^2\right)\phi(\mathbf{r}) = 0 \text{ in } r > R \ (R < L/2)$$

from Klein-Gordon eq. of two-particle (in c.m. frame)

2. Finite volume method

Helmholtz equation on L^3

$$(\nabla^2 + p^2)\phi(\mathbf{r}) = 0, \quad p^2 = (E^2 - 4m^2)/4 \neq (2\pi/L \cdot \mathbf{n})^2$$

1. Solution in $r > R$

$$\phi(\mathbf{r}) = C \cdot \sum_{\mathbf{n}\in Z^3} \frac{e^{i\mathbf{r}\cdot\mathbf{n}(2\pi/L)}}{\mathbf{n}^2 - q^2}, \quad q^2 = \left(\frac{Lp}{2\pi}\right)^2 \neq n$$

2. Expansion by spherical Bessel $j_l(pr)$ and Noeman $n_l(pr)$ functions

$$\phi(\mathbf{r}) = \beta_0(p) n_0(pr) + \alpha_0(p) j_0(pr) + (l \geq 1)$$

3. S-wave Scattering phase shift $\delta_0(p)$ in infinite volume

$$\tan\delta_0(p) = \frac{\beta_0(p)}{\alpha_0(p)} = \frac{\pi^{3/2} q}{Z_{00}(1;q^2)}$$

$$Z_{00}(1;q^2) = \frac{1}{\sqrt{4\pi}} \sum_{\mathbf{n}\in Z^3} \frac{1}{\mathbf{n}^2 - q^2}$$

Note: $\phi(\mathbf{r})$ can be calculated from lattice QCD.

$\rightarrow$ Sasaki's talk, Hatsuda's talk and Nemura's talk

2. Finite volume method

δ_0 from $E^2 = 4(m^2 + p^2)$

$$\tan\delta_0(p) = \frac{\pi^{3/2} q}{Z_{00}(1;q^2)}, \qquad q^2 = \left(\frac{Lp}{2\pi}\right)^2$$

Lüscher NPB354 531(1991)

Expand around $p \approx 0$ ($a_0 = \lim_{p\to 0} \delta_0(p)/p$)

$$E_0 - 2m = -\frac{4\pi a_0}{mL^3}\left(1 + c_1 \frac{a_0}{L} + c_2 \frac{a_0^2}{L^2}\right) + O(1/L^6)$$

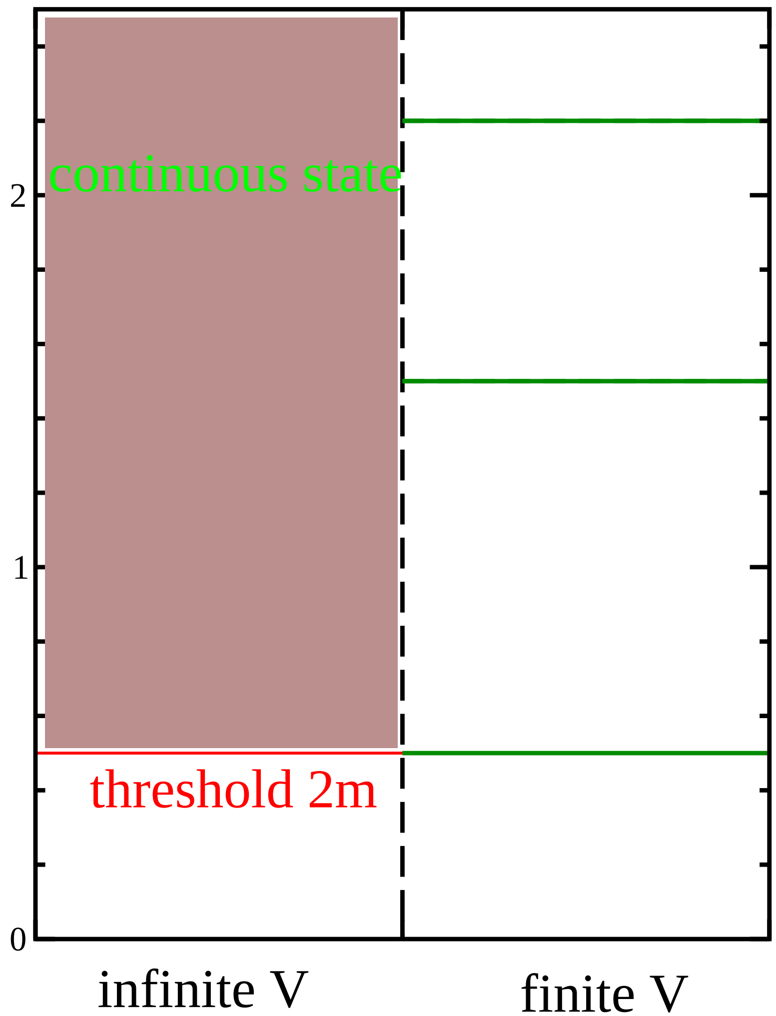
$c_1 = -2.837297, c_2 = 6.375183$ Lüscher CMP105 153(1986)

$\Delta E = E_n - E_n^{\text{free}}$; $E_n^{\text{free}} = 2\sqrt{m^2 + (2\pi/L \cdot \mathbf{n}^2)}$

Sign of ΔE depends on interaction

ΔE	interaction	$\tan\delta_0$ or a_0
> 0	repulsive	< 0
< 0	attractive	> 0

Scattering state



Continuous state in infinite volume

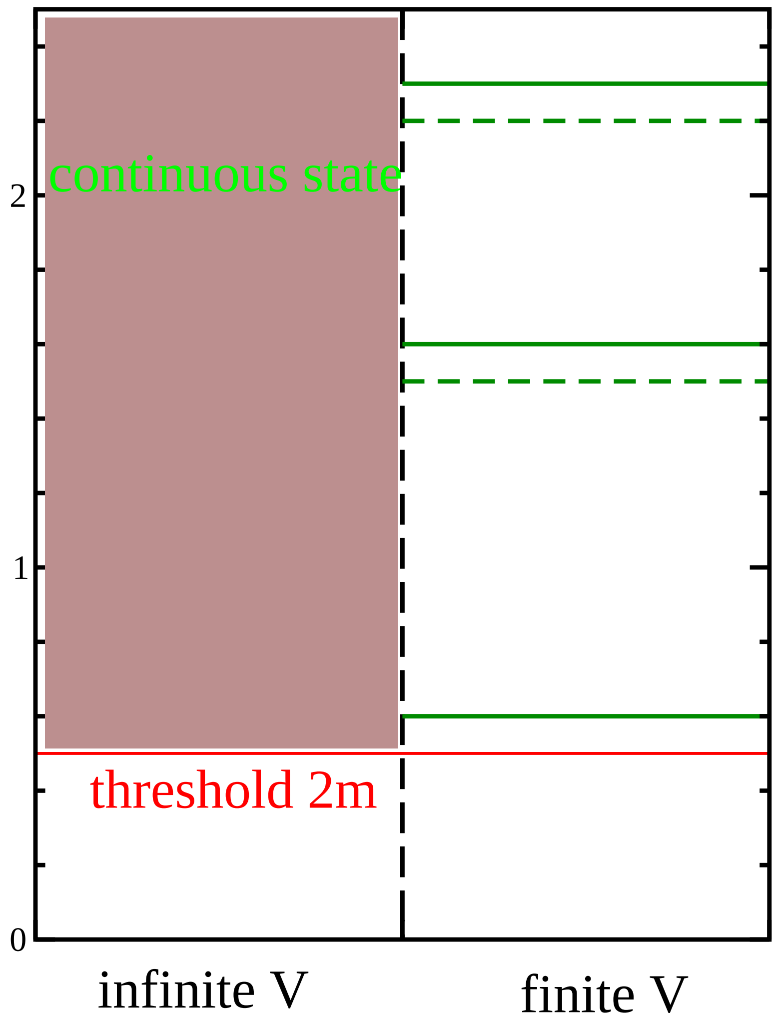
$\longrightarrow$

discrete state in finite volume

No interaction case

$$E^2 = 4(m^2 + (2\pi/L \cdot \mathbf{n})^2)$$

Scattering state



Continuous state in infinite volume

$\longrightarrow$

discrete state in finite volume

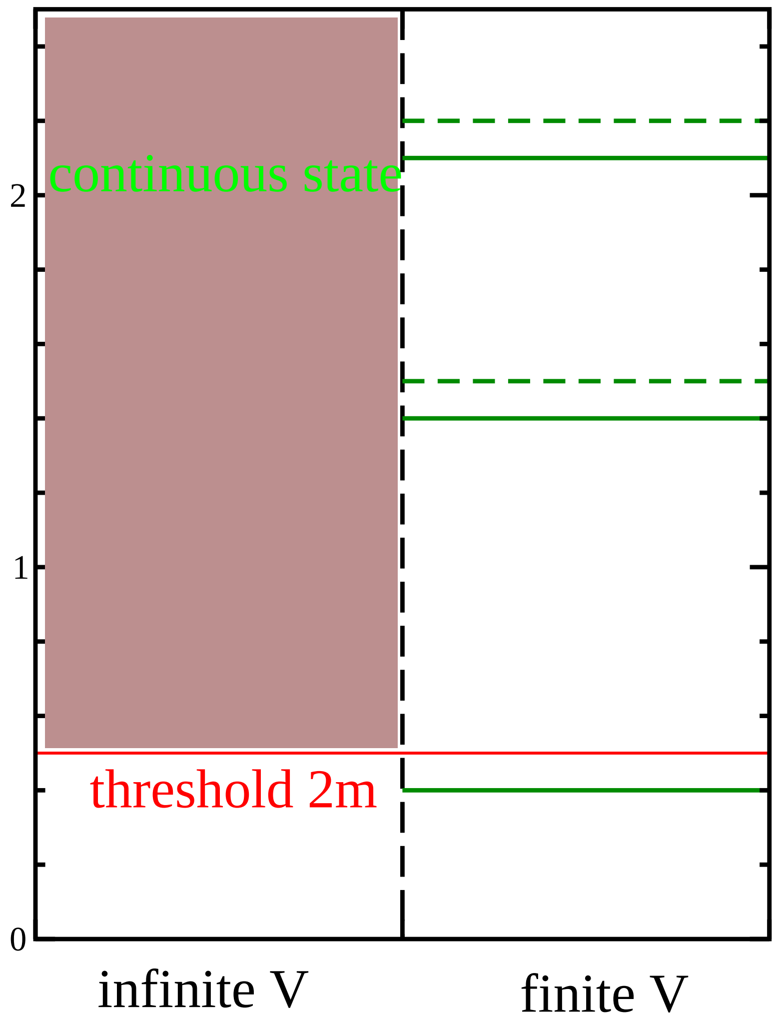
No interaction case

$$E^2 = 4(m^2 + (2\pi/L \cdot \mathbf{n})^2)$$

Repulsive interaction $a_0 < 0$

$\Delta E > 0$

Scattering state



Continuous state in infinite volume

$\longrightarrow$

discrete state in finite volume

No interaction case

$$E^2 = 4(m^2 + (2\pi/L \cdot \mathbf{n})^2)$$

Repulsive interaction $a_0 < 0$

$\Delta E > 0$

Attractive interaction $a_0 > 0$

$\Delta E < 0$

Scattering states

Previous works of scattering state

1. $I = 2$ $\pi\pi$ channel (a_0 and δ_0)

'92	Sharpe, Gupta and Kilcup	'04	CP-PACS Collaboration
'93	Gupta, Patel and Sharpe		Du, Meng, Miao and Liu
	Kuramashi *et al.*	'05	CP-PACS Collaboration
'99	JLQCD Collaboration		BGR Collaboration
'02	Liu, Zhang, Chen and Ma		NPLQCD Collaboration
	JLQCD Collaboration	'07	CLQCD Collaboration
'03	BGR Collaboration	'08	NPLQCD Collaboration
	CP-PACS Collaboration		Sasaki, Ishizuka
	Kim		Yamazaki

2. $I = 3/2$ $(1/2)$ $K\pi$ channel $\longrightarrow$ Muroya's talk

'04	Miao, Du, Meng, Liu	'08	Nagata, Muroya, Nakamura
'06	NPLQCD	on going	Sasaki, Ishizuka

3. $I = 1$ KK channel

'08 NPLQCD

Previous works of scattering state (cont'd)

4. $(\eta_c, J/\psi) - (\pi, \rho, n)$ channel

   '06 Yokokawa, Sasaki, Hatsuda, Hayashigaki   '08 Liu, Lin, Orginos

5. ${}^1S_0$, ${}^3S_1$ nn channel $\longrightarrow$ Hatsuda's talk

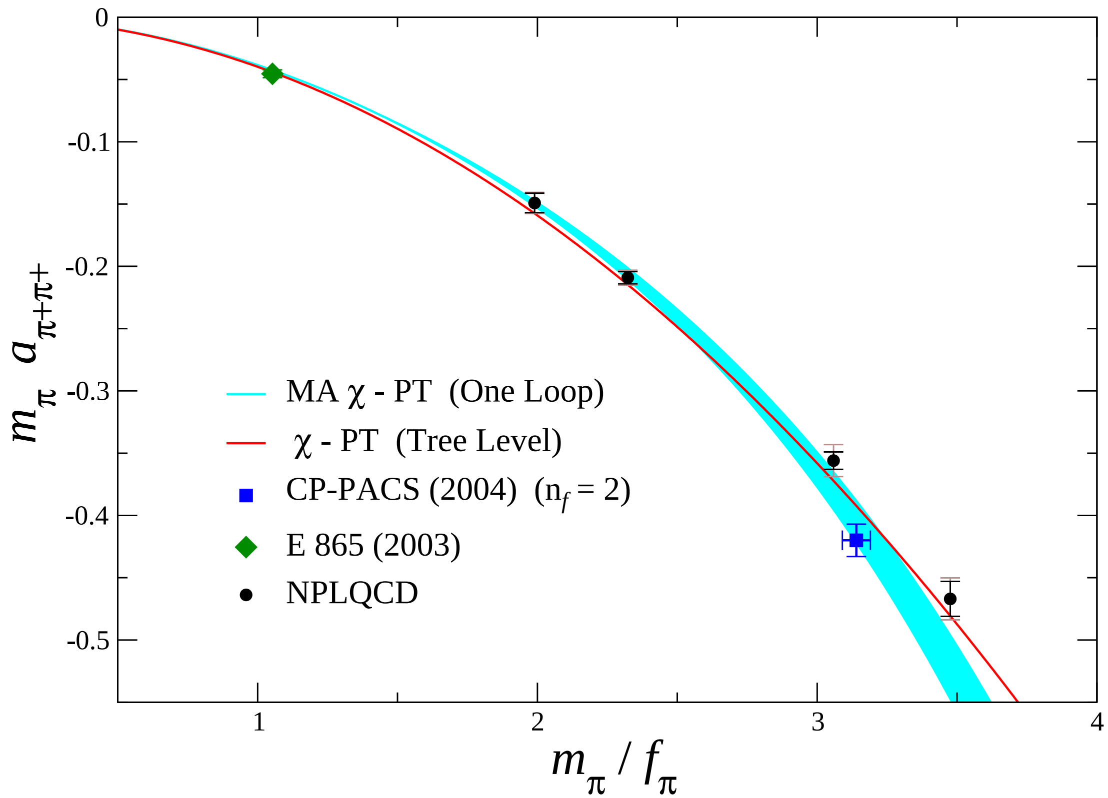
   '95 Fukugita, Kuramashi, Okawa, Miho, Ukawa   '07 Ishii, Aoki, Hatsuda
   '06 Beane, Bedaque, Orginos, Savage

6. $n\Sigma$, $n\Lambda$, $n\Xi$ channel $\longrightarrow$ Nemura's talk

   '07 NPLQCD   '08 Nemura, Ishii, Aoki, Hatsuda

πn, Kn, $\cdots$ etc.

S-wave $I = 2$ $\pi\pi$ a_0 in $1/a = 1.6$ GeV
with DWF valence on $N_f = 2 + 1$ imp-staggered sea
'08 NPLQCD Collaboration



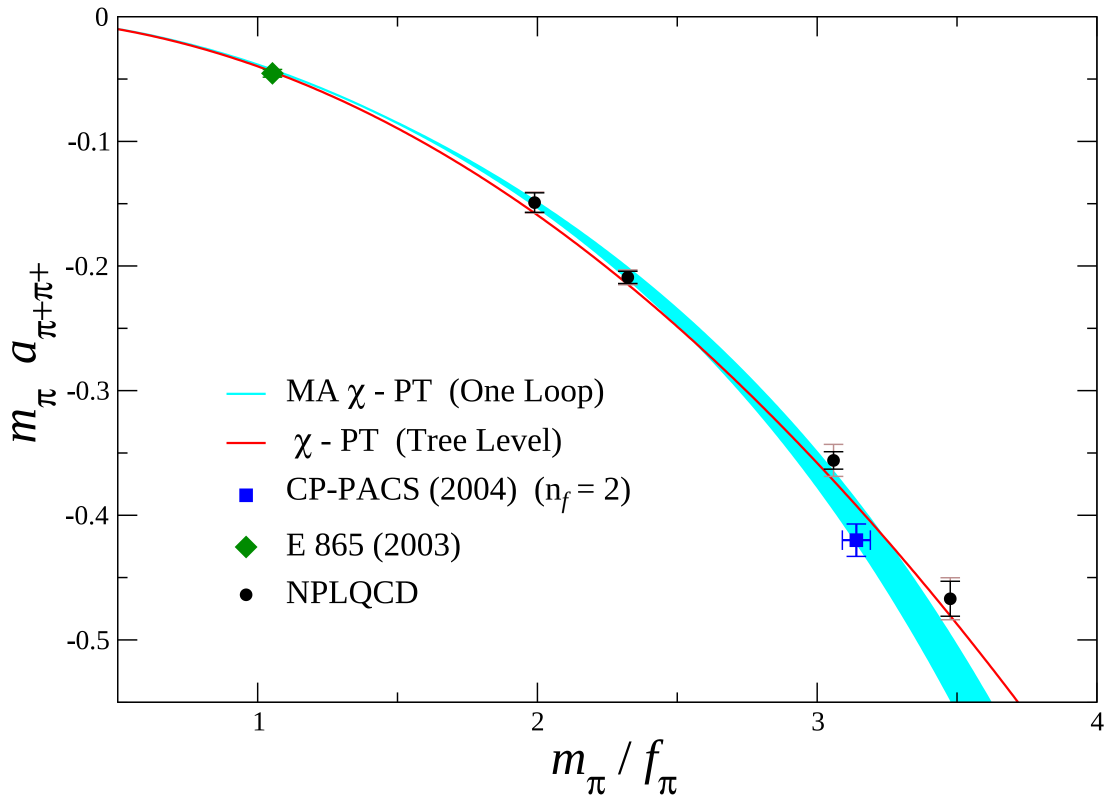
Result is almost consistent with LO ChPT.

$$a_0 m_\pi \;=\; -\frac{m_\pi^2}{8 f_\pi^2}\left\{1 + \frac{m_\pi^2}{16\pi^2 f_\pi^2}\left[3\ln\left(m_\pi^2/\mu^2\right) - l(\mu) - \frac{\Delta_{ju}^4}{6 m_\pi^4}\right]\right\}$$

$$\Delta_{ju}^4 \equiv 2B_0(m_{\text{sea}}^j - m_{\text{val}}^u) + a^2\Delta_I + \cdots$$

S-wave $I = 2$ $\pi\pi$ a_0 in $1/a = 1.6$ GeV with DWF valence on $N_f = 2 + 1$ imp-staggered sea

'08 NPLQCD Collaboration



NPLQCD $a_0 m_\pi = -0.04330(42)$

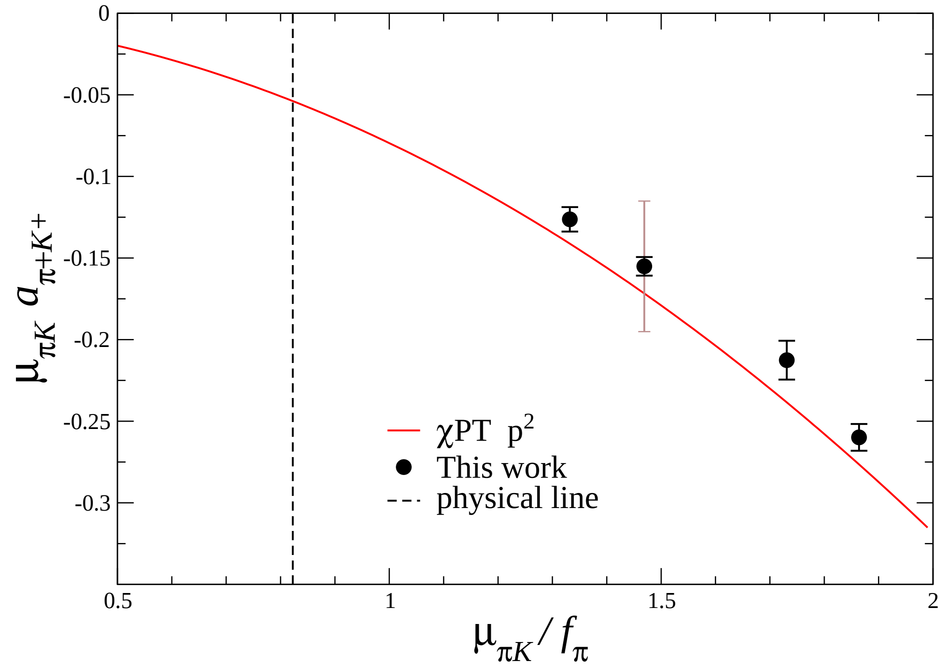
ChPT (2-loop) $a_0 m_\pi = -0.0444(10)$

Exp. $a_0 m_\pi = -0.0454(33)$

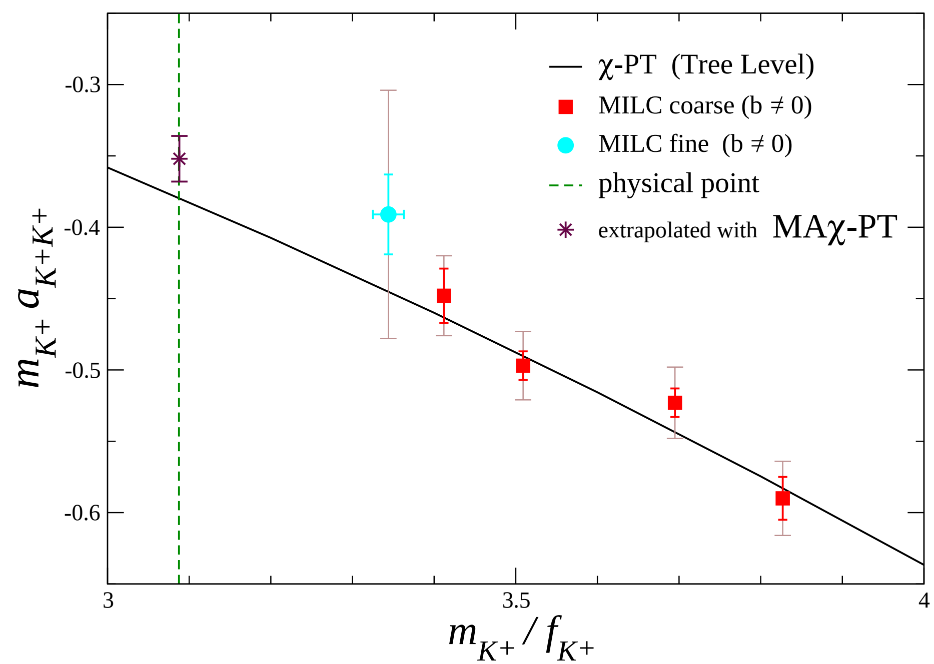
S-wave a_0 of $I = 3/2\ K\pi$ and $I = 1\ KK$ in $1/a = 1.6$ GeV with DWF valence on $N_f = 2+1$ imp-staggered sea

'06 '08 NPLQCD Collaboration

$I = 3/2\ K\pi$



$I = 1\ KK$



$$a_0^{3/2} m_\pi = -0.0574(44)$$

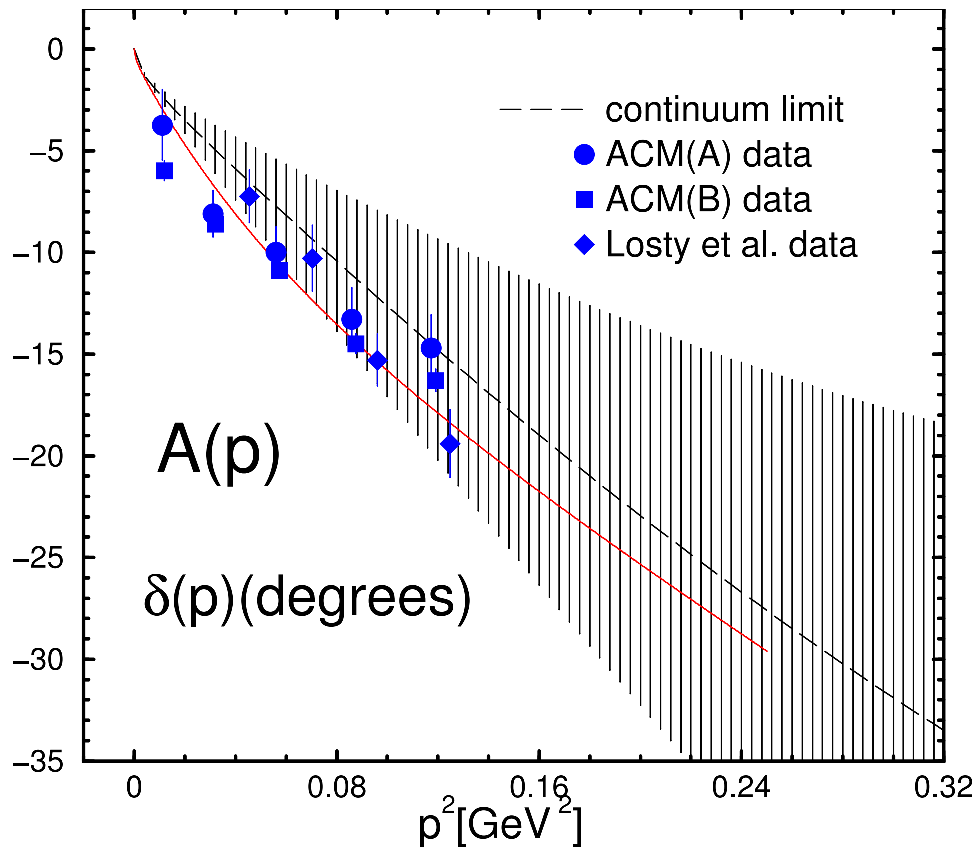
$$a_0^{1} m_K = -0.0352(16)$$

$$a_0^{1/2} m_\pi = 0.1725(90)$$ through NLO ChPT

Result is roughly consistent with LO ChPT.

S-wave $I = 2$ $\pi\pi$ $\delta(p)$ in $a = 0$ with $N_f = 2$ imp-Wilson

'04 CP-PACS Collaboration



The symbols are the experimental data. (Hoogland *et al.*, Losty*et al.*)

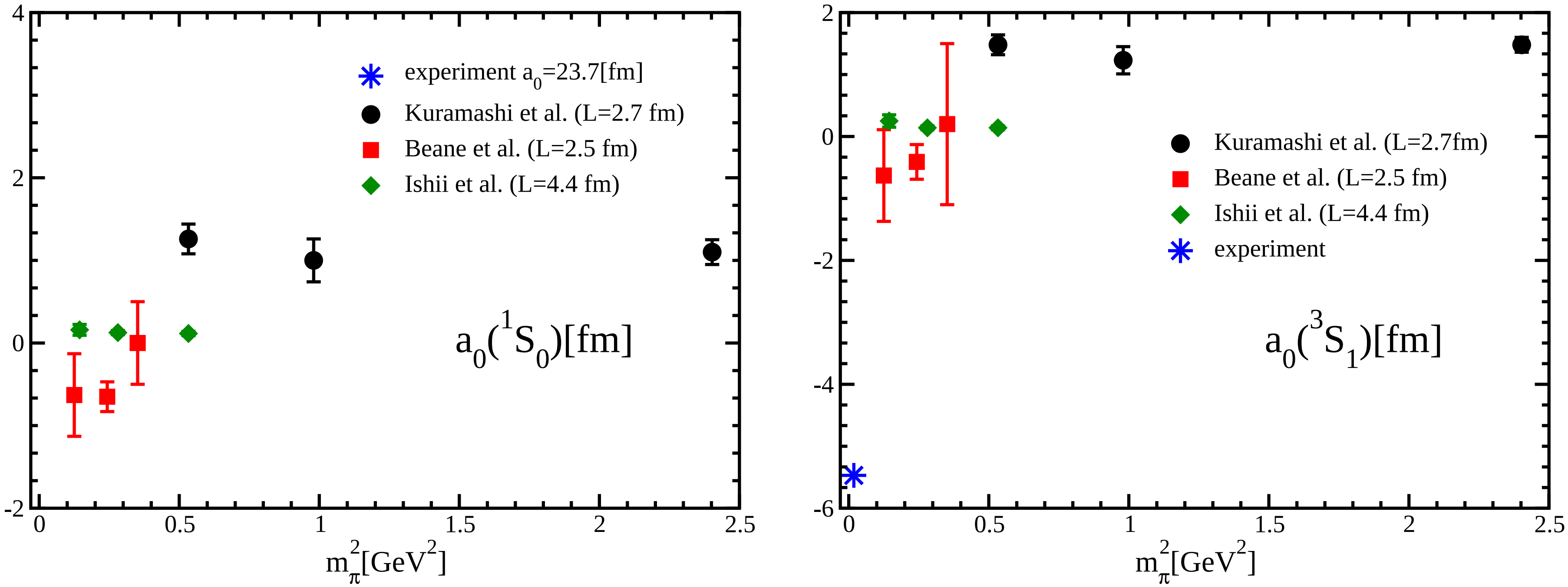
The solid line is parametrized by experimental input. (Colangelo *et al.*)

Result is almost consistent with experimental data.

Large error from $a \to 0$ extrapolation

S-wave a_0 in $^1S_0$ and $^3S_1$ nn

'95 Kuramashi et al.
'06 Beane et al.
'07 Ishii et al.



Action and β are same in two works, but volume is different.

Lightest m_π of Kuramashi *et al.* = Heaviest m_π of Ishii *et al.*

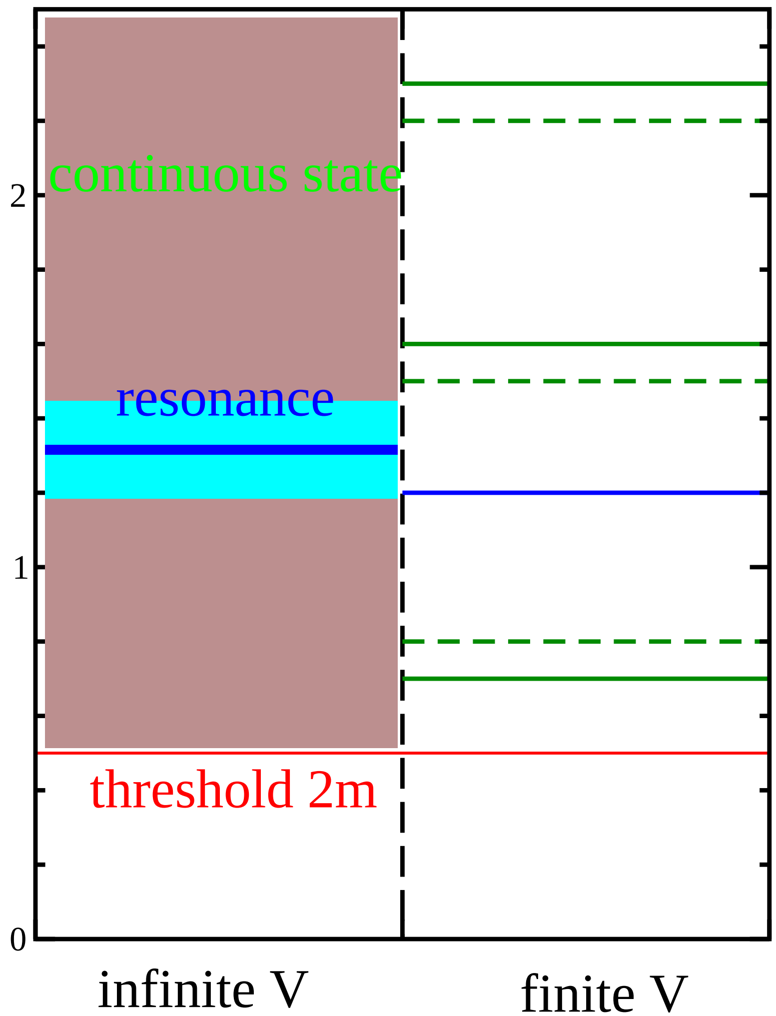
Large finite volume effect?

need detail study of systematic errors

lighter pion mass, larger volume, other systematic error?

Resonance state

Scattering + resonance states



Resonance state with Γ in infinite volume
$\longrightarrow$
discrete state in finite volume

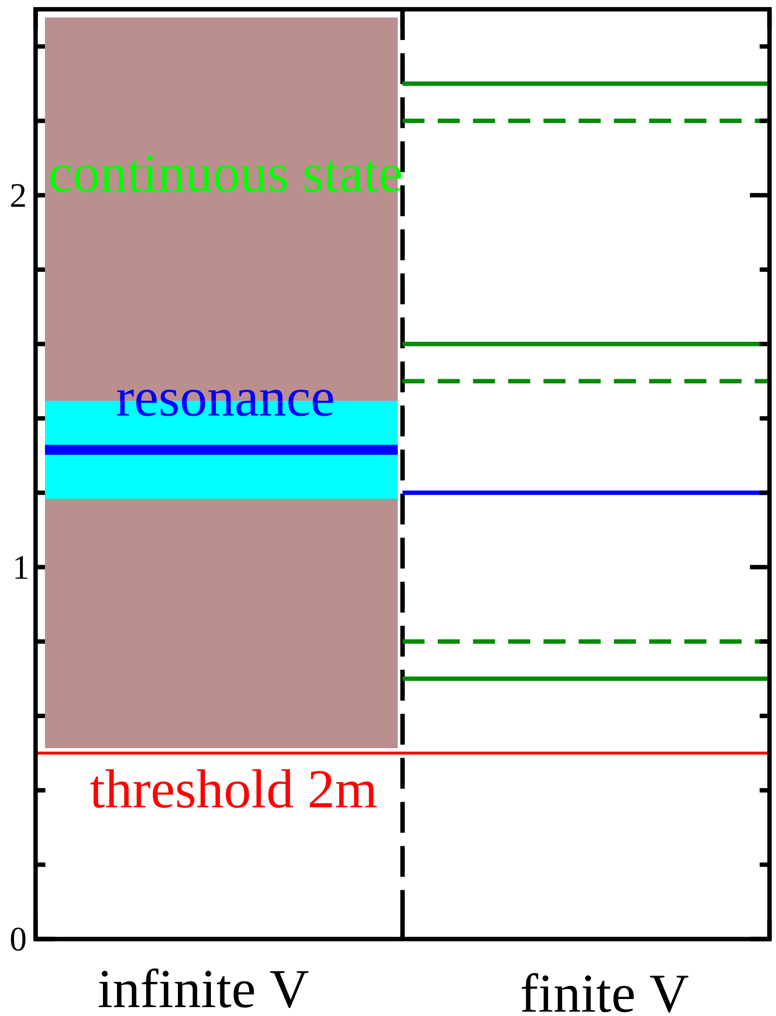
Energy is always real in finite volume.
$\rightarrow$ Γ does not appear in E

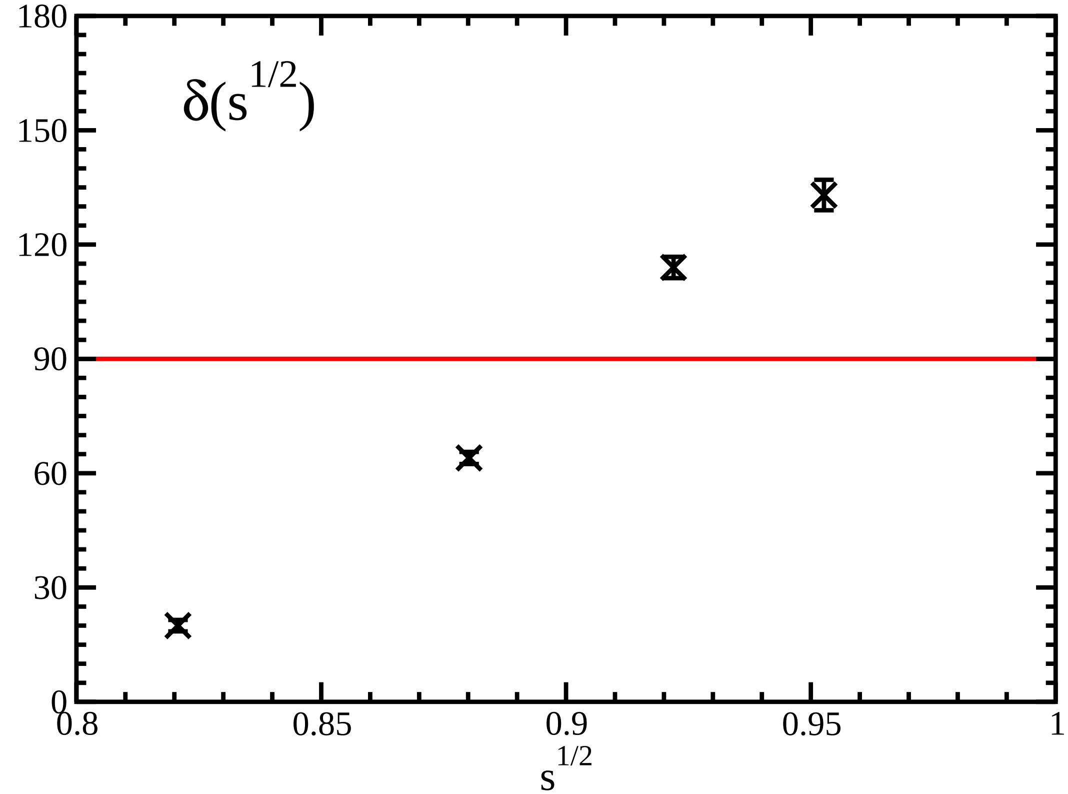
Energy shift in finite volume

$\tan\delta_0 < 0 \rightarrow \Delta E > 0$

$\tan\delta_0 > 0 \rightarrow \Delta E < 0$

Scattering + resonance states

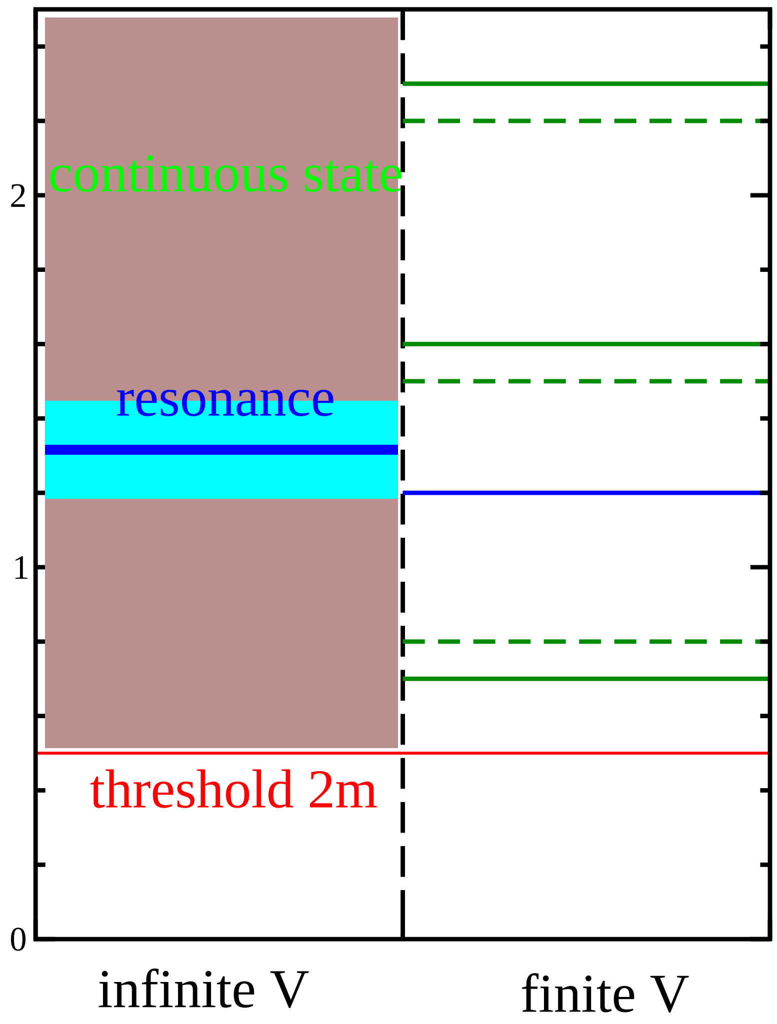


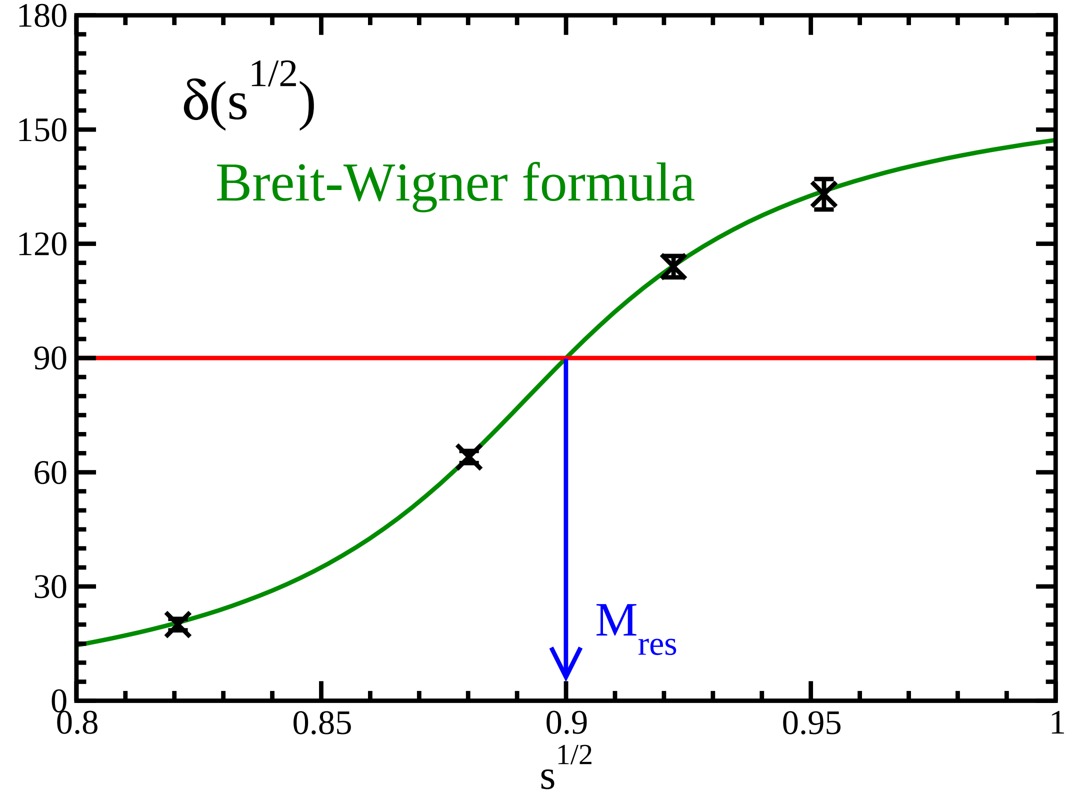


$\delta(\sqrt{s})$ in infinite volume

from E on finite volume

Scattering + resonance states '91 Lüscher; '94 Göckeler *et al.*





$$\cot \delta(M_{\text{res}}) = 0 \Rightarrow \delta(M_{\text{res}}) = \pi/2$$

$$\Gamma = \tan \delta(\sqrt{s}) \frac{M_{\text{res}}^2 - s}{M_{\text{res}}} \bigg|_{\sqrt{s}=M_{\text{res}}}$$

Previous works of resonance state

- $I = 1\ \pi\pi \to \rho$ channel

'84	Gottlieb, Machenzie, Thacker, Weingarten
'89	Loft, DeGrand
'03	McNeile, Michael
'07	CP-PACS Collaboration
'08	Bernard, Meißner, Rusetsky (proposed method)

1. Decay amplitude ($m_\pi / m_\rho > 0.5$)

$$\langle 0|\rho(t)\pi\pi(0)|0\rangle \sim \left[1 + t\langle\rho|\pi\pi\rangle + O(t^2)\right] \times e^{-m_\rho t}$$

2. δ from E $\to m_\rho$ and Γ

3. Discrete probability distribution ($\pi n \to \Delta$)

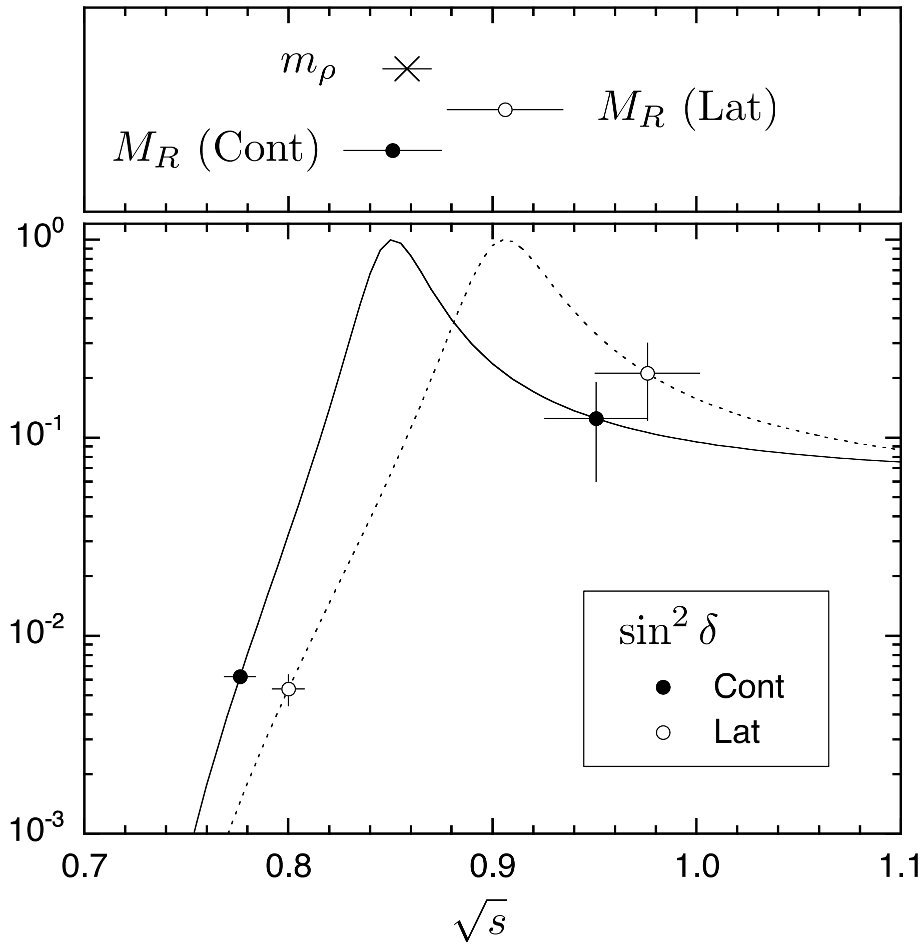
need huge number of data (volume and momentum)

- $S = 1\ Kn \to \Theta^+$ channel

'03	Csikor, Fodor, Katz, Kovacs	'05	Ishii, Doi, Nemoto, Oka, Suganuma
'04	Sasaki		Lassock *et al.*
'05	Takahashi, Umeda, Onogi, Kunihiro	'06	Alexandrou, Tsapalis
	Okiharu, Suganuma, Takahashi		Csikor, Fodor, Katz, Kovacs, Toth
			...

P-wave $I = 1$ $\pi\pi$ $\delta(p)$ at $m_\pi = 0.33$ GeV with $N_f = 2$ full QCD

'07 CP-PACS Collaboration



Breit-Wigner form

$$\tan\delta(p) = \frac{g^2_{\rho\pi\pi}}{6\pi}\frac{p^3}{\sqrt{s}(m_\rho^2 - s)}$$

$g_{\rho\pi\pi} = 6.25(67)$

$m_\rho a = 0.851(24)$

At physical point

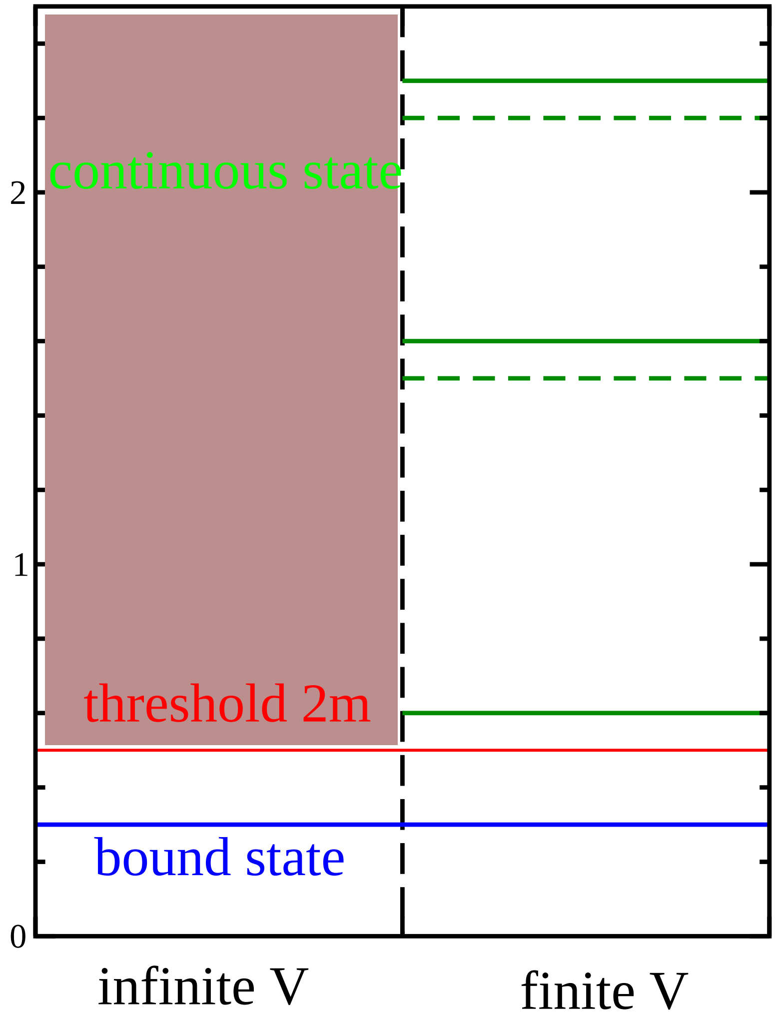
$$\Gamma = \frac{g^2_{\rho\pi\pi}}{6\pi}\frac{(p_\rho^{\rm ph})^3}{(m_\rho^{\rm ph})^2} = 162(35)\ \text{MeV}$$

$$(p_\rho^{\rm ph})^2 = ((m_\rho^{\rm ph})^2 - 4(m_\pi^{\rm ph})^2)/4$$

$\Gamma_{\rm exp} = 150$ MeV

Bound state

Scattering + bound states



Continuous state in infinite volume

$\longrightarrow$

discrete state in finite volume

Positive binding energy

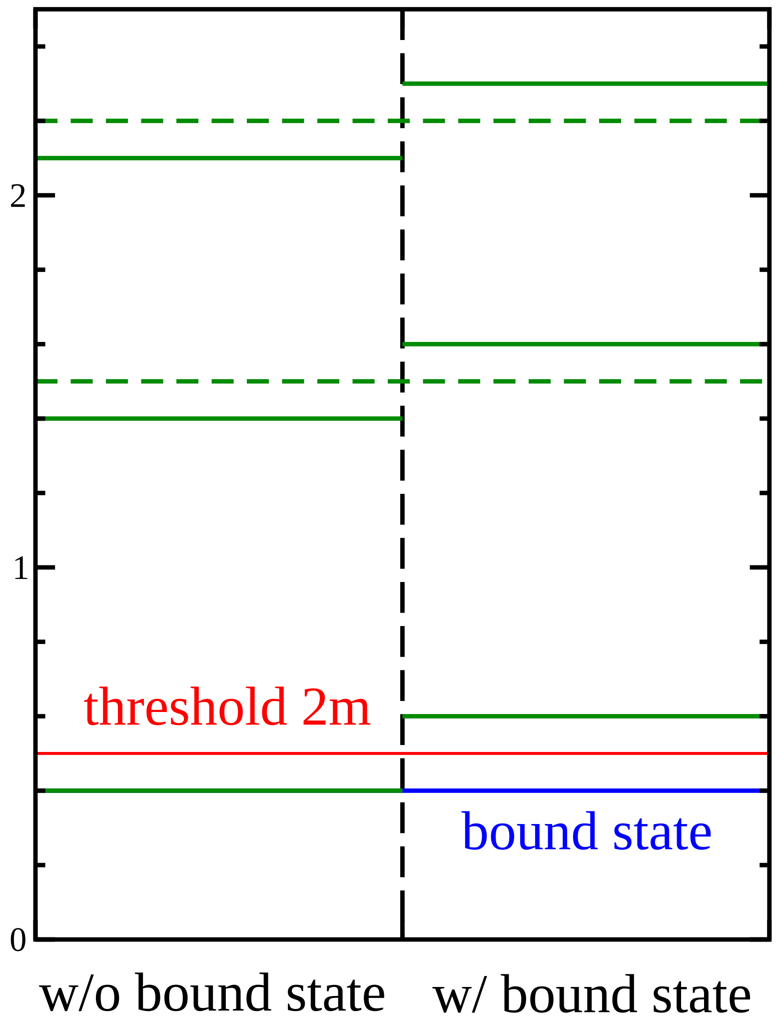
$$\Delta E_B = 2m - M_B$$

Bound state isolated in infinite volume

$\longrightarrow$

one of discrete states in finite volume

Scattering + bound states

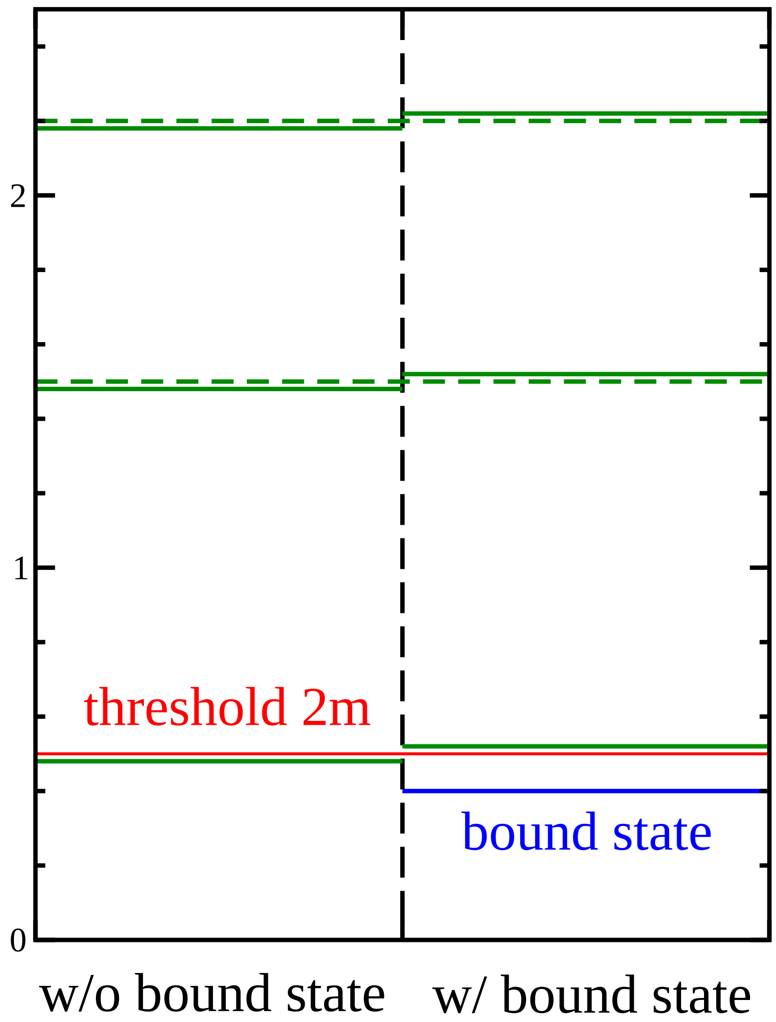


Problem

If $\Delta E_B \approx 0$, like dueteron $\Delta E_B = 2$ MeV, hard to identify bound state in finite volume

Similar energy shift ($\Delta E < 0$) in attractive system

Scattering + bound states



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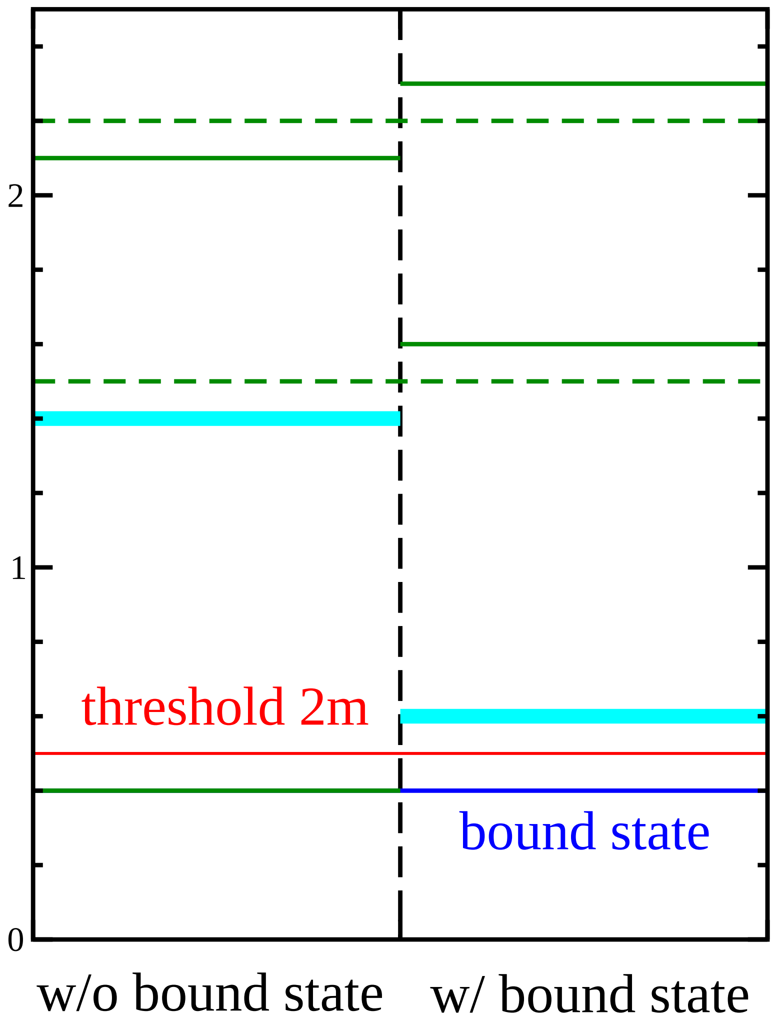
Similar energy shift ($\Delta E < 0$) in attractive system

Solutions

1. Volume dependence
$\Delta E_B \neq 0$ but $\Delta E_0 \to 0$ in $L \to \infty$
'04 Beane *et al.*

Scattering + bound states



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Solutions

1. Volume dependence
$\Delta E_B \neq 0$ but $\Delta E_0 \to 0$ in $L \to \infty$
'04 Beane *et al.*

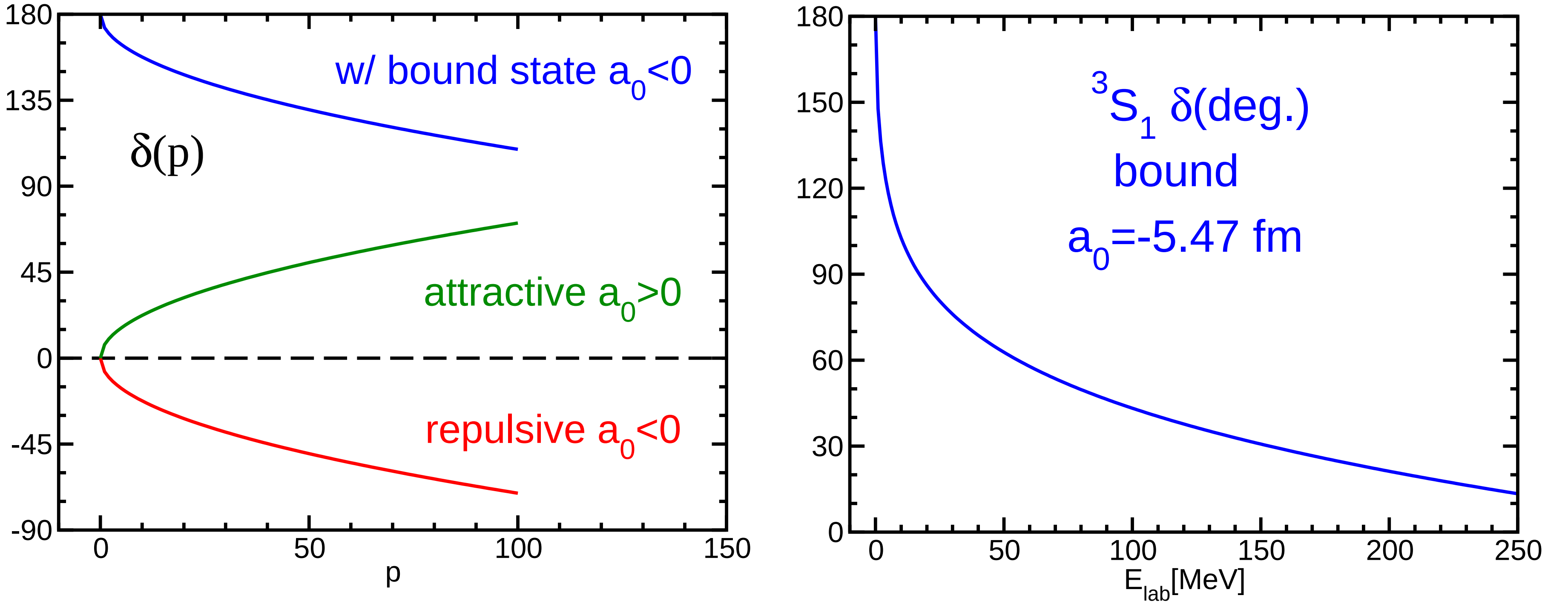
2. First excited state energy

Scattering + bound states

$a_0 < 0$ when one bound state exists.

Levinson's theorem

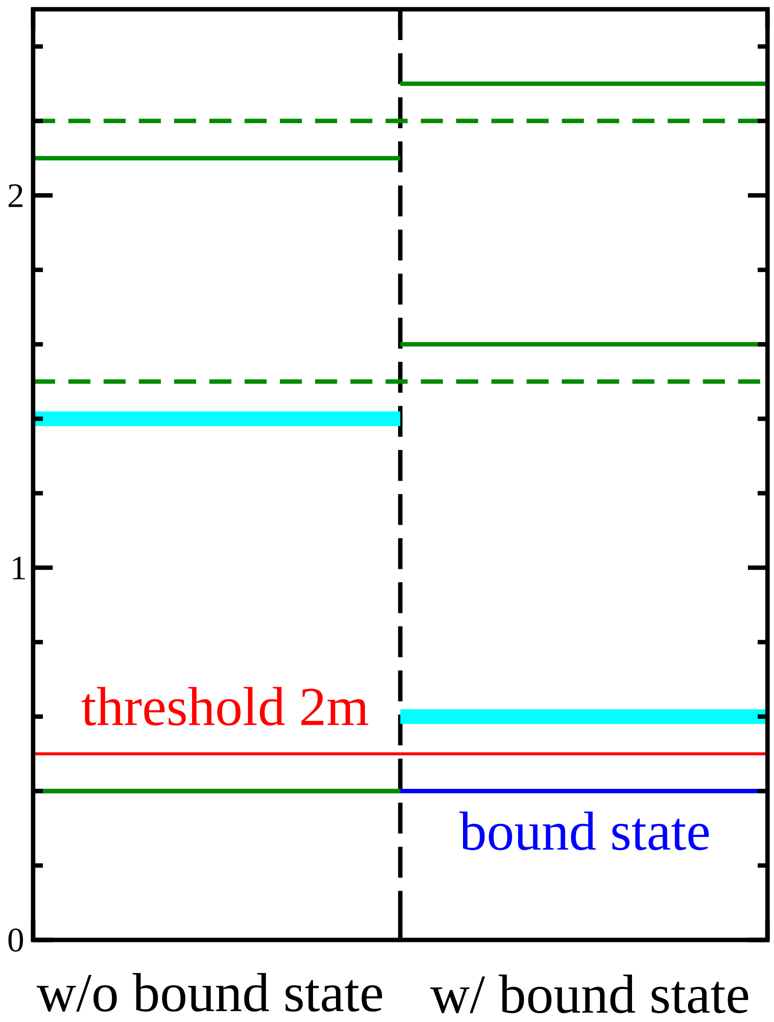
$$\delta(0) = n\pi \quad (n = \# \text{ of bound state})$$



Plots from NN-OnLine (http://nn-online.org/)

$np\ ^3S_1$ scattering has negative scattering length.

Scattering + bound states



Problem

If $\Delta E_B \approx 0$, like dueteron $\Delta E_B = 2$ MeV, hard to identify bound state in finite volume

Similar energy shift ($\Delta E < 0$) in attractive system

Solutions

1. Volume dependence
$\Delta E_B \neq 0$ but $\Delta E_0 \to 0$ in $L \to \infty$
'04 Beane *et al.*

2. First excited state energy
$E_1 \sim 2m$ and $\Delta E_1 > 0$ but
$E_1 \sim 4(m^2 + p_1^2)$
'06 Sasaki and TY

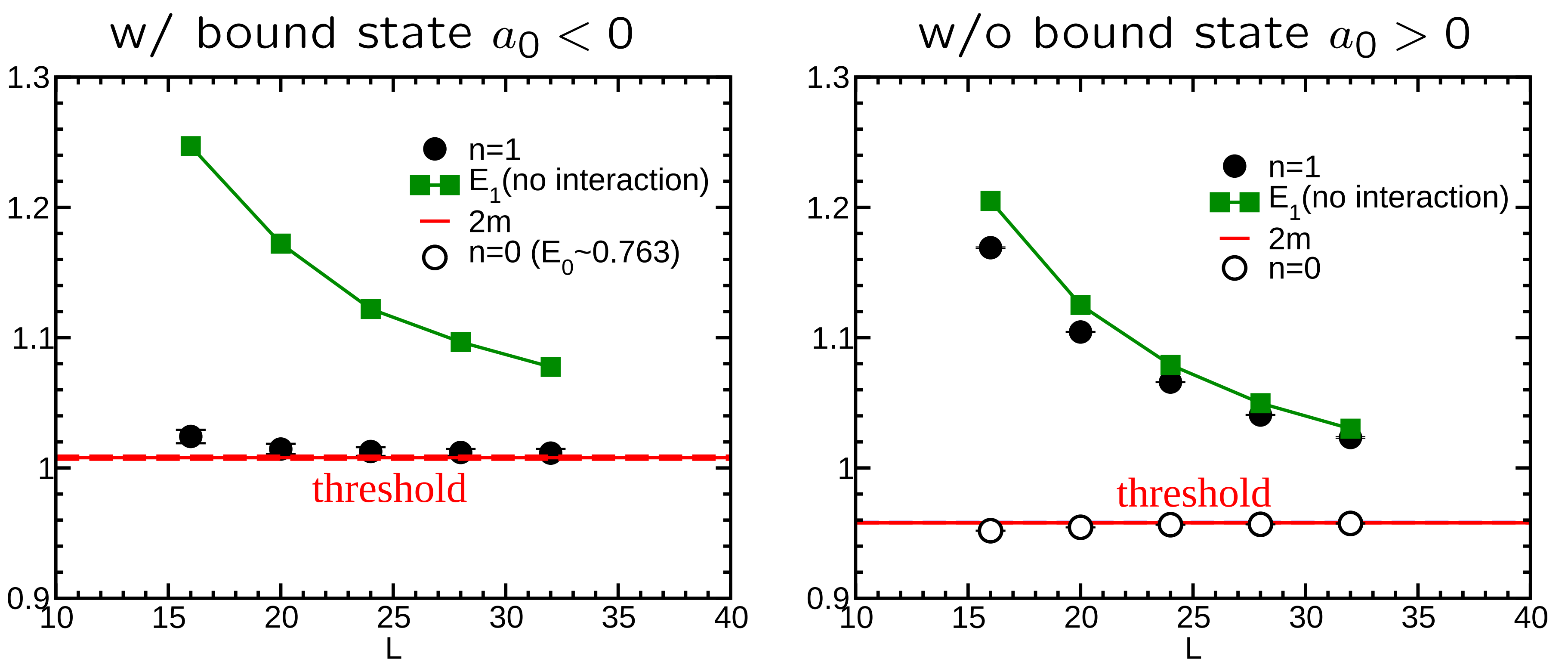
Two-particle energy on finite volume

'06 Sasaki and TY

Effective model QED (Abelian-Higgs) with Scalar field $|\Phi_x| = 1$

Easy to control bound state formation with charge Q of fermion

Large binding energy case $\Delta E_B = 2m - M_B \sim 1.0 - 0.76 = 0.24$



Two-particle energy on finite volume

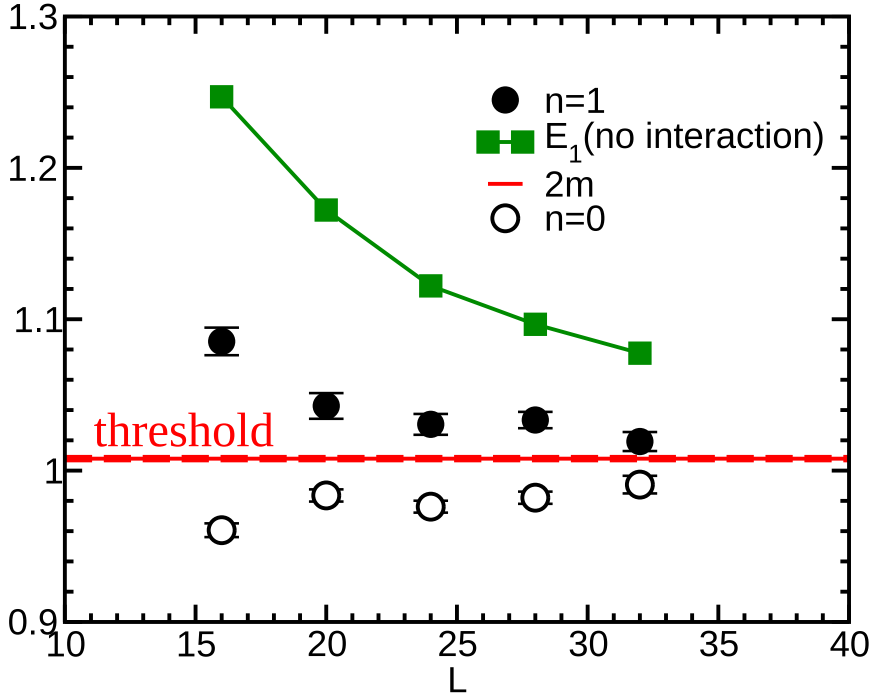
'06 Sasaki and TY

Effective model QED (Abelian-Higgs) with Scalar field $|\Phi_x| = 1$

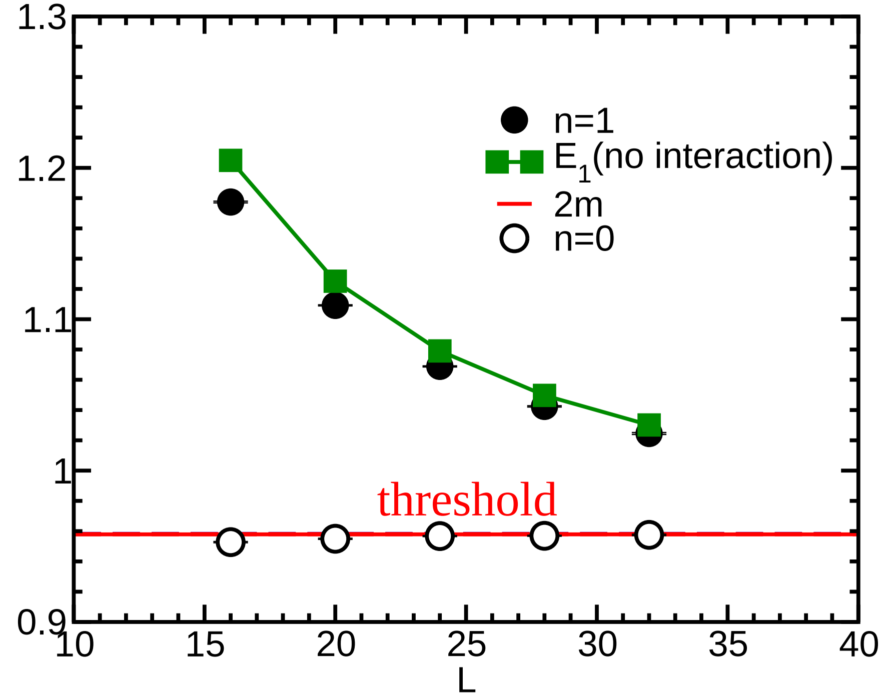
Easy to control bound state formation with charge Q of fermion

Small binding energy case $\Delta E_B = 2m - M_B \sim 1.0 - 0.99 = 0.01$

w/ bound state $a_0 < 0$



w/o bound state $a_0 > 0$

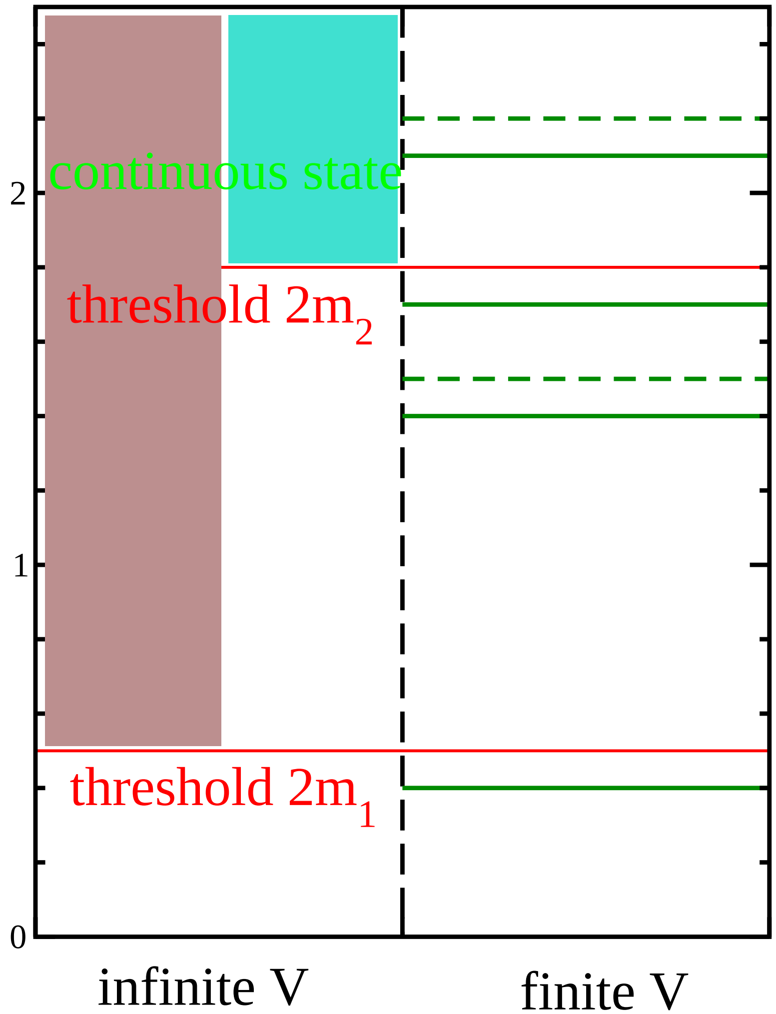


Significant difference in first excited state.

$\Rightarrow$ Signal of bound state formation

More complex scattering states

Scattering + scattering states (S-wave $I = 0$ $\pi\pi$ + KK)



Continuous state in infinite volume

$\longrightarrow$

discrete state in finite volume

$2m_\pi < E < 2m_K$

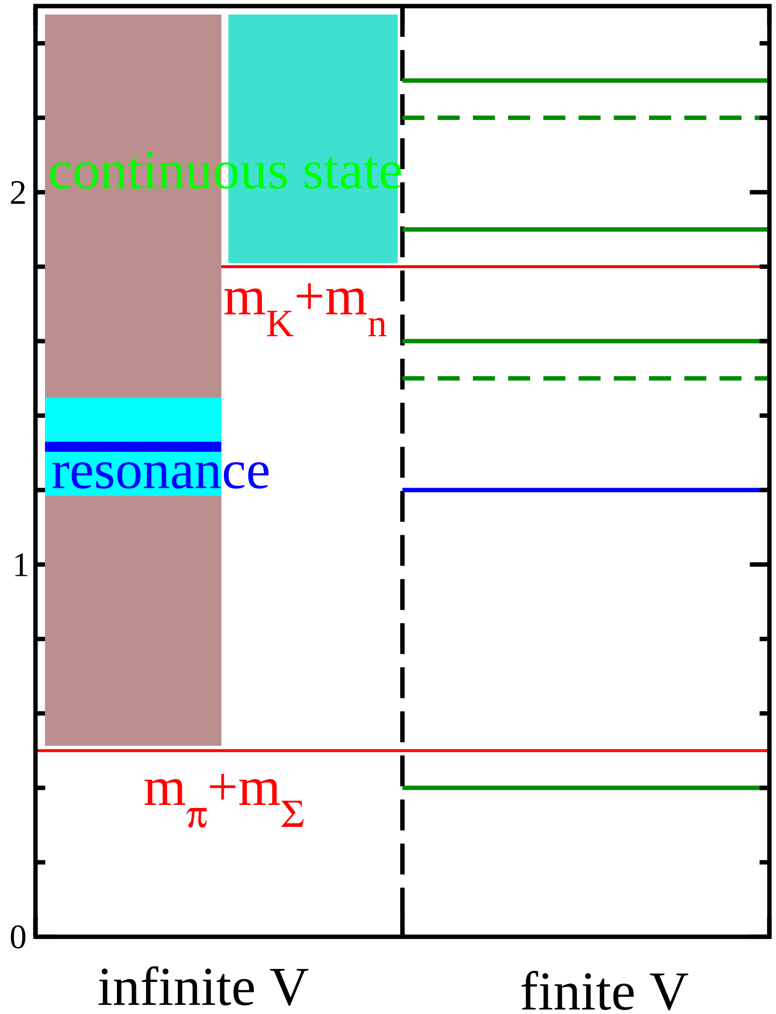
$E \leftrightarrow \delta(E)$

$2m_K < E$

$E \leftrightarrow \delta_{\pi\pi}(E), \delta_{KK}(E), \eta(E)$

'05 He, Feng, Liu

Scattering + resonance + scattering states (S-wave $I = 0$ $\pi\Sigma$ + $\Lambda(1405)$ + Kn)



Continuous state in infinite volume

$\longrightarrow$

discrete state in finite volume

$m_\pi + m_\Sigma < E < m_K + m_n$

$$E \leftrightarrow \delta(E)$$

At least we can determine

a_0, $m_{\Lambda(1405)}$, $\Gamma_{\Lambda(1405)}$

$m_K + m_n < E$

Too complicated

further study needed

Summary

Scattering state

a_0 : $\pi\pi$, KK, πK in highest I channel comparing with ChPT

δ_0 : $\pi\pi$ in $I = 2$ with $N_f = 2$ at $a = 0$

a_0 : nn in ${}^1S_0$ and ${}^3S_1$

large systematic uncertainty $\rightarrow$ detail study needed

Resonance state

δ : P-wave $I = 1$ $\pi\pi \rightarrow \rho$

Breit-Wigner form $\rightarrow$ m_ρ and Γ

Bound state

Problem of dueteron $\Delta E_D = 2$ MeV

Method to identify bound state formation

Volume dependence and first excited state energy

More complex scattering

S-wave $I = 0$ $\pi\Sigma + \Lambda(1405) + Kn$ too complicated

At least we could obtain $a_0^{\pi\Sigma}, \delta_{\pi\Sigma}$ with present knowledge

Multi-meson scattering state $\rightarrow$ $\mu_{K^-} = \left.\frac{dE_{nK}}{dn}\right|_{V=\text{const.}}$

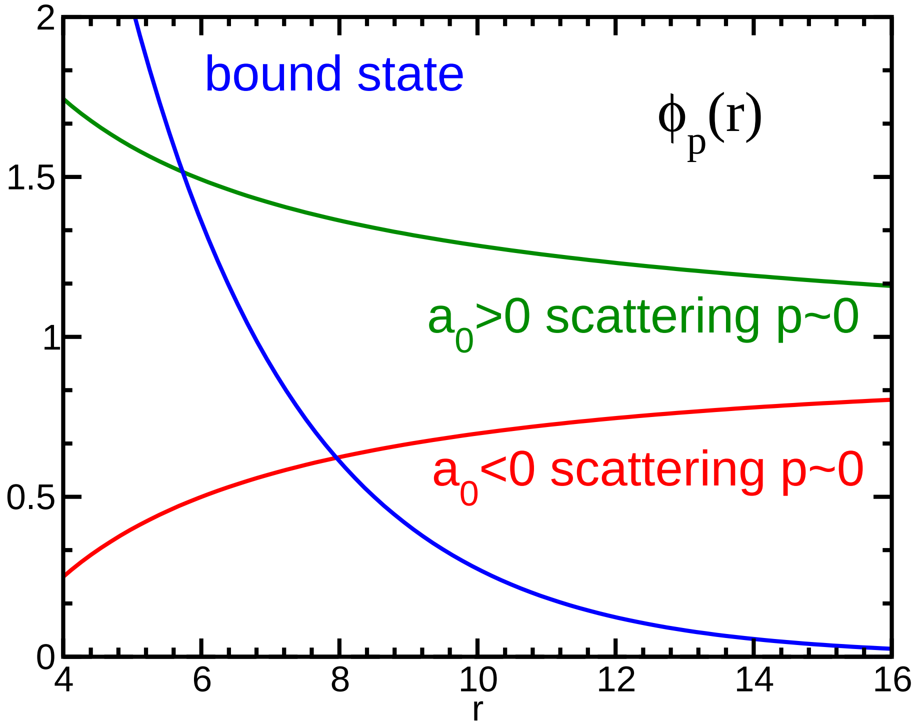
Backup Slides

Two-particle wave function on finite volume

Balog *et al.*, RRD60:094508(1999); CP-PACS, PRD71:094504(2005); Aoki *et al.*, PRL99:022001(2007); ....

S-wave wave function in large r region

$$\phi_p(\vec{r}) = \langle 0|f(r)f(0)|ff;p\rangle \quad (p^2 = E^2/4 - m^2)$$
$$\sim \frac{\sin(pr + \delta(p))}{pr} \quad \text{(scattering state)}$$
$$\sim \exp(-\gamma r) \quad \text{(bound state)}$$



Large difference in $\phi_p(\vec{r})$

bound state $\leftrightarrow$ scattering state

scattering state: $a_0 > 0 \leftrightarrow a_0 < 0$

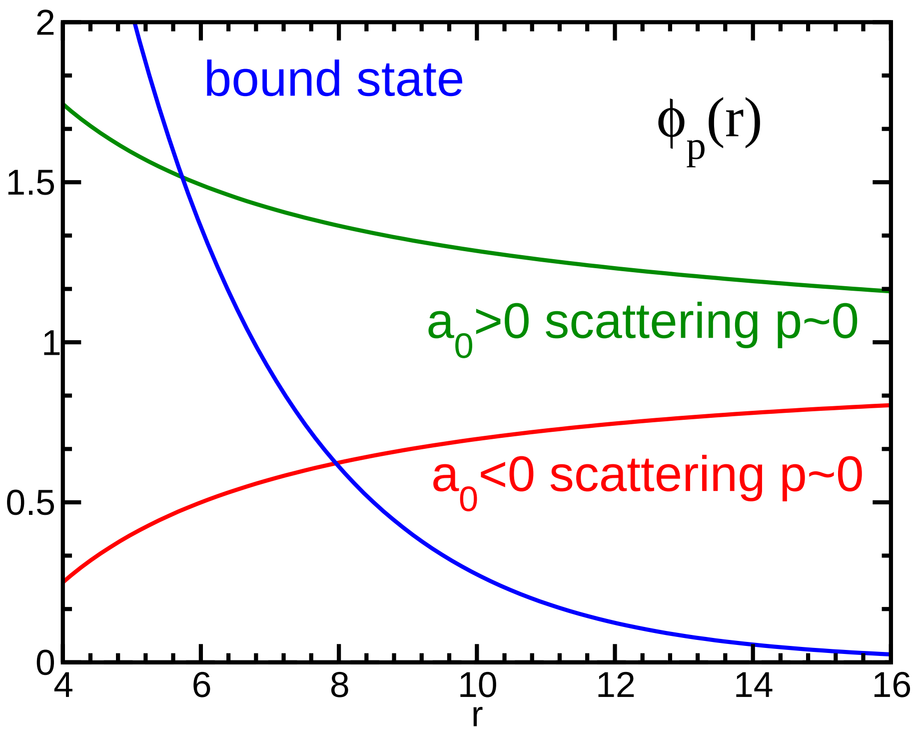
Two-particle wave function on finite volume

Effective model

QED (Abelian-Higgs) with Scalar field $|\Phi_x| = 1$

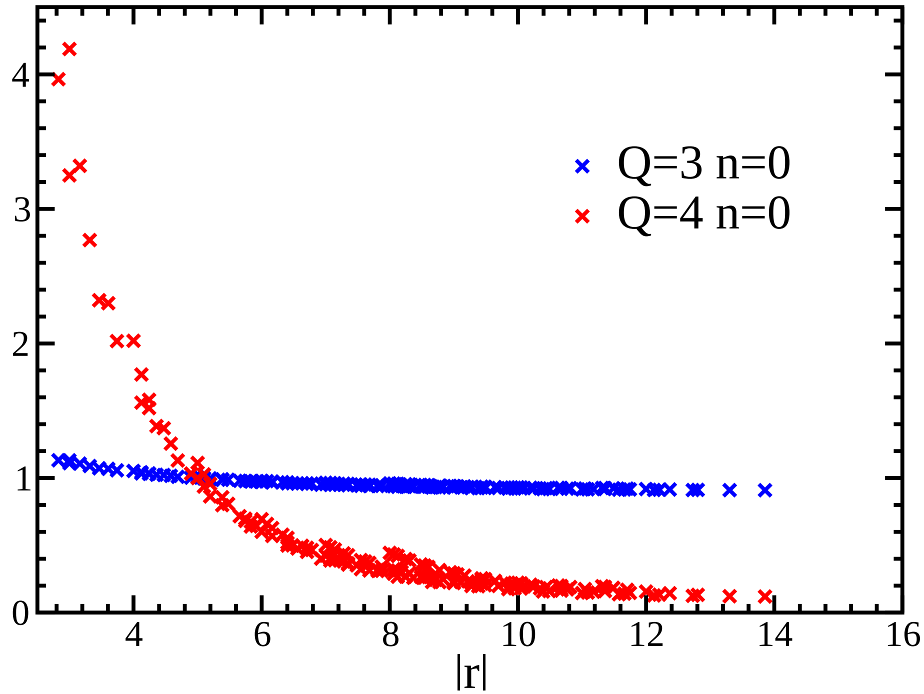
Easy to control bound state formation with charge Q of fermion

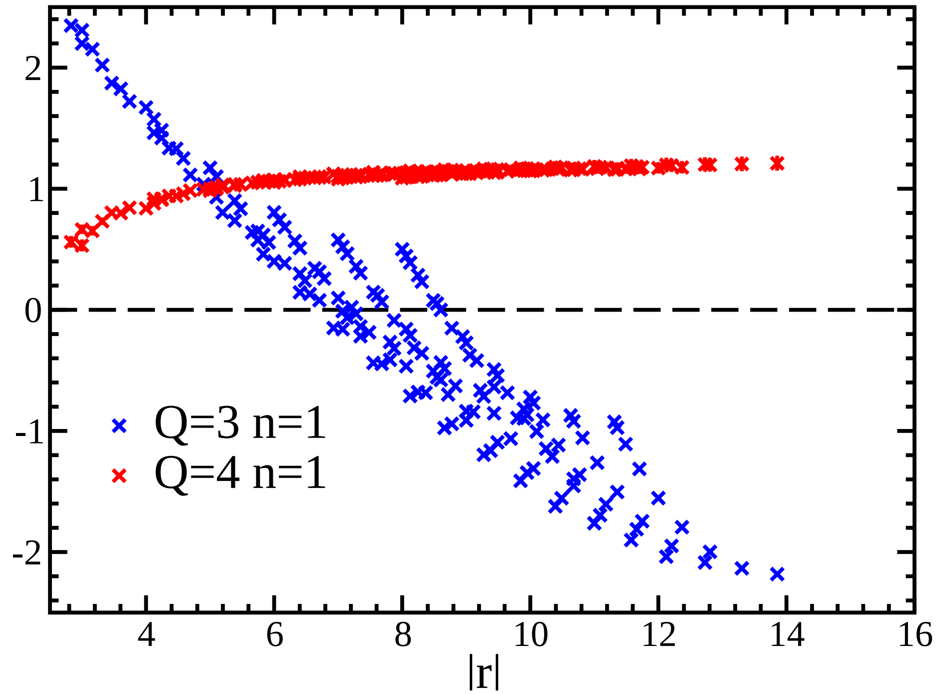
Calculate $\phi_p(\vec{r})$ of lower two states of S-wave $\overline{f}f$ state, like meson, with different charge $Q = 3, 4$



sys.	a_0	$n = 0$	$n = 1$
w/o bound	$+$	$p \sim 0$	$p \sim 2\pi/L$
w/ bound	$-$	bound	$p \sim 0$

Wave function $\phi_p(\vec{r})/\phi_p(r=5)$ $(Q=3,4)$





Different $\phi_p(\vec{r})$ for each state as expected

sys.	a_0	$n=0$	$n=1$
w/o bound	$+$	$p \sim 0$	$p \sim 2\pi/L$
w/ bound	$-$	bound	$p \sim 0$

Consistent with $M - 2m = 0.75 - 2 \times 0.5 \approx -0.25 (Q=4)$

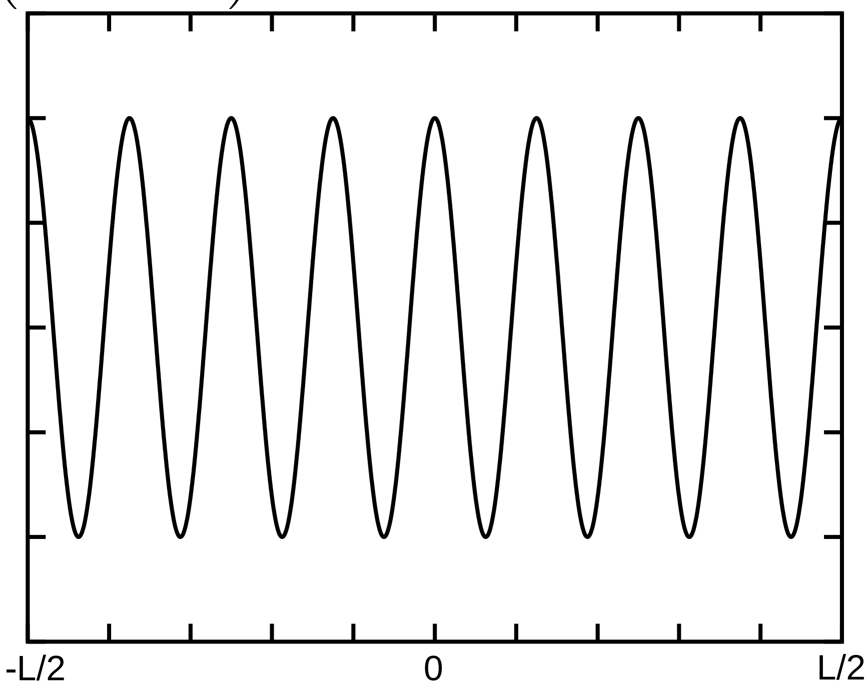
Wave function might be a candidate to identify bound state.

One-dimension L case with periodic boundary condition

Two-pion wave function $\Phi(r)$ satisfies periodic boundary condition.

Free case

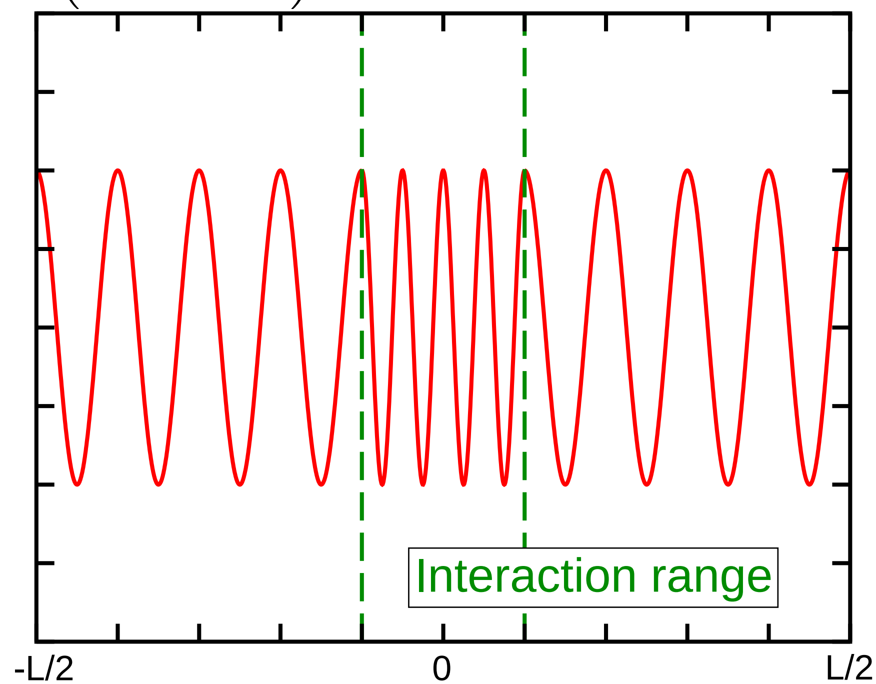
$\left(\nabla^2 + p_0^2\right)\Phi(r) = 0$



$p_0^2 = (2\pi/L)^2 \cdot n^2$, n is integer.

Interacting case

$\left(\nabla^2 + p^2\right)\Phi(r) = V_p(r)\Phi(r)$



$V_p(r) \neq 0$ in $r < R$

$p, \delta(p), L$,p.b.c. $\rightarrow e^{i2\delta(p)+pL} = 1$

$p^2 = (2\pi/L)^2 \cdot q^2$, q^2 is not integer.

p^2 has information of $\delta(p)$.

2.2 Diagonalization method

Problem on lattice (c.m. frame)

pion four-point function

$$
\begin{aligned}
G_{nm}(t) &= \langle 0|\Omega_n(t)\Omega_m(0)|0\rangle, \quad \Omega_i = \pi(\mathbf{p}_n)\pi(-\mathbf{p}_m) \\
&= \sum_\alpha V_{\alpha n} V^t_{\alpha m} e^{-E_\alpha t}, \; V_{\alpha n} = \langle \overline{\pi\pi}_\alpha|\Omega_n|0\rangle \\
&\to V_{0n} V^t_{0m} e^{-E_0 t}, \quad t \to \infty
\end{aligned}
$$

Ω_n : two-pion operator ($p_n^2 = (2\pi/L \cdot \mathbf{n})^2$)

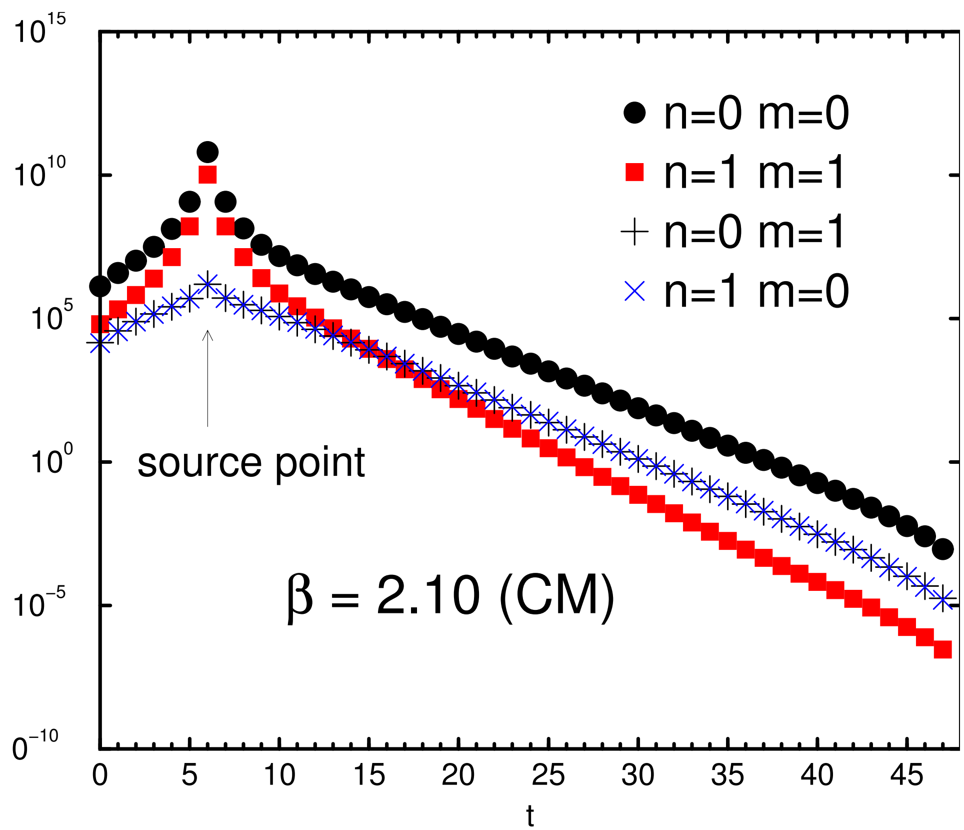
$\langle\overline{\pi\pi}_\alpha|$: α-th two-pion state

Four-point function behaves a multiexponential form.

We cannot obtain $E_{\alpha (\alpha \neq 0)}$ by single exponential fit.

2.2 Diagonalization method

pion four-point function $G_{nm}(t)$ '04 CP-PACS Collaboration



$p_0 = (0,0,0)$, $p_1 = (1,0,0)$

All G_{nm} have same slope in $t \gg 1$.

Diagonalization of four-point function matrix

Lüscher and Wolff, Nucl. Phys. B339 222(1990)

pion four-point function matrix

$$G_{nm}(t) = \langle 0|\Omega_n(t)\Omega_m(0)|0\rangle$$

$$M(t,t_0)w_\nu = \lambda_\nu(t,t_0)w_\nu$$

$$\lambda_\nu(t,t_0) = \exp\left(-E_\nu(t-t_0)\right)$$

$M(t,t_0) = G^{-1}(t_0)G(t)$ with t_0 : reference point

We can extract $E_{\nu\neq 0}$ from $\lambda_\nu(t,t_0)$.

In actual calculation $G(t)$: $N \times N$ matrix

assumption : $N+1$-th state contribution is negligible

Other methods

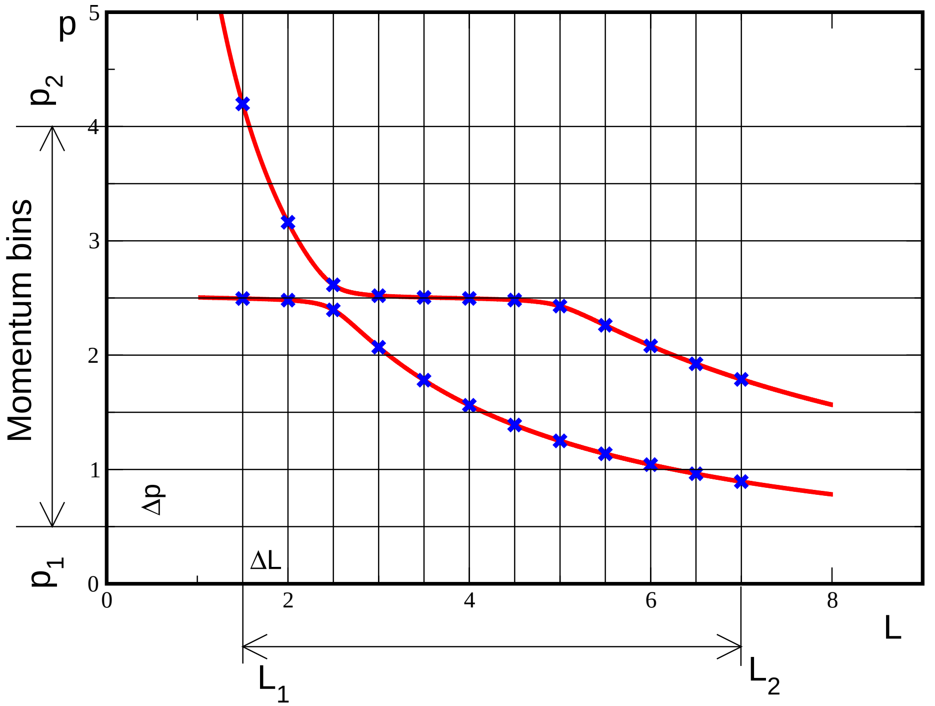
Anti-periodic boundary condition '04 Kim

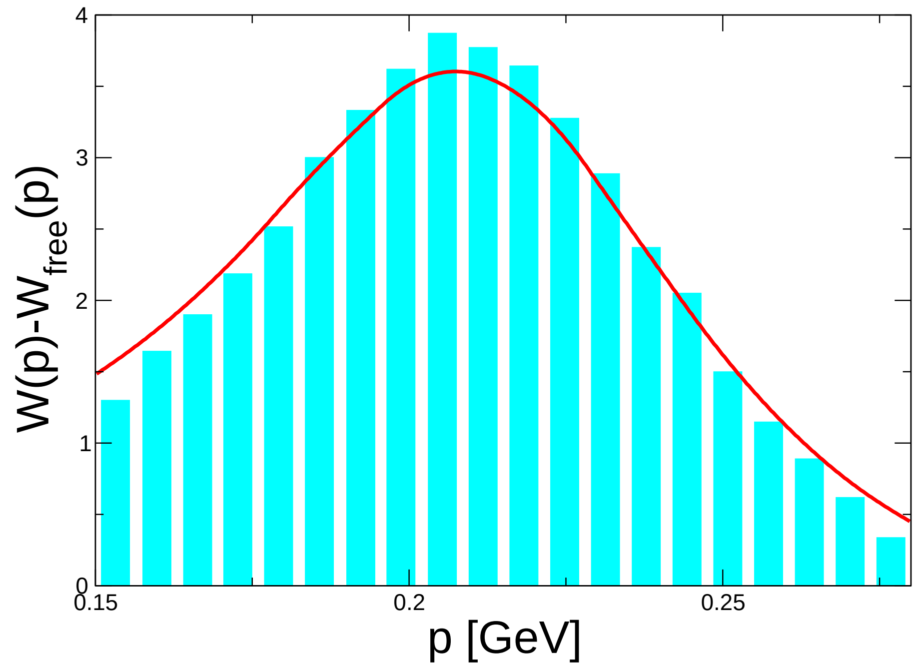
Non-zero total momentum $\pi(\mathbf{p}_1) \longrightarrow \leftarrow \pi(\mathbf{p}_2)$ '95 Rummukainen and Gottlieb

...

Discrete probability distribution ($\pi n \to \Delta$)

'08 Bernard, Meißner, Rusetsky





No resonance parametrization of infinite volume scattering phase

$W(p)$: # of state in a fixed p bin
$W_{\text{free}}(p)$: # of state in a fixed p bin in free case

$$m_\Delta \text{ from } W(p) - W_{\text{free}}(p)$$