# Higher Dimensional Black Holes

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# Purpose of this talk

– give a brief overview of recent progress in understanding basic properties of $D \geqslant 4$ black holes

# Black holes in general relativity

**Definitions:**

Consider a strongly causal, asymptotically flat spacetime $(M, g_{ab})$
Let $\mathcal{I}^+$ be a set of *idealized* distant observers (i.e., future *null infinity*)

Black hole $B := M -$ Chronological past of $\mathcal{I}^+$

Event horizon of $B$ $\mathcal{H} :=$ Boundary of $B$

Predictable $B$ $:\Leftrightarrow$ Outside $B$ (including $\mathcal{H}$) is globally hyperbolic

**Remarks:**

- *$\mathcal{H}$ is, by definition, a null hypersurface*
- *$\mathcal{H}$ is a global notion; it has no distinguished local significance*

**Area Theorem:** (Hawking 71)

Consider a predictable black hole that is a solution to Einstein's equation with the null energy conditions

The surface area $A$ of horizon cross-sections of $\mathcal{H}$ can never decrease with time

The null energy conditions: $T_{ab}k^a k^a \geqslant 0$ for any null vectors $k^a$

**Remark:**

*A resemblance to $2$nd-law of thermodynamics:*

(Entropy $S$ never decreases: $\delta S \geqslant 0$) (Bekenstein 73)

# Towards local characterization of black holes

- Gravitating source bends the spacetime geometry

$$ds^2 = -(1-2\Phi)dt^2 + \frac{dr^2}{1-2\Phi} + r^2 d\Omega^2 \ : \ -\Phi : \text{ Newton potential}$$

Static observers (along $t^a = (\partial/\partial t)^a$) are no longer geodeisc but are accelerated: i.e.,

$$t^c \nabla_c t^a = \kappa(r)(\partial/\partial r)^a$$

- A distinguished null hypersurface $\mathcal{N}$ in the limit $\Phi \to 1/2$ since $g^{ab}(dr)_a(dr)_b \to 0$, $t^a t_a \to 0$ as $\Phi \to 1/2$ ($r \to r_H$)

$$t^c \nabla_c t^a = \kappa t^a \quad \text{on} \quad \mathcal{N}$$

– $t^a$ is *tangent* and *normal* to $\mathcal{N}$

– $\kappa = \kappa(r_H)$: *redshifted proper acceleration* of observer $t^a$ on $\mathcal{N}$

# Killing Horizons

**Definitions:**

Killing horizon $\mathcal{N}$ :⇔ A null hypersurface whose null generators coincide with the orbits of a one-parameter group of isometries (so that there is a Killing field $K^a$ *normal* to $\mathcal{N}$)

Surface gravity $\kappa$ of $\mathcal{N}$ :⇔ A function on $\mathcal{N}$ that satisfies

$$\nabla^a(K^b K_b) = -2\kappa K^a \quad \cdots\cdots (*)$$

**Remarks:**

– *eq. ($*$) is rewritten as* $K^b \nabla_b K^a = \kappa K^a$

– *The notion of a Killing horizon is independent of the notion of event horizon*

– *Surface gravity $\kappa$ must be constant along each null generator of $\mathcal{N}$, but can, in general, vary from generator to generator*

In a wide variety of circumstances, the event horizon $\mathcal{H}$ of a stationary black hole is also a Killing horizon

e.g., Kerr metric (rotating black hole)

$$\begin{aligned} \mathrm{d}s^2 &= \rho^2\left(\frac{\mathrm{d}r^2}{\Delta}+\mathrm{d}\theta^2\right)+(r^2+a^2)\sin^2\theta\mathrm{d}\phi^2 \\ &\quad -\mathrm{d}t^2+\frac{2Mr}{\rho^2}(a\sin^2\theta\mathrm{d}\phi-\mathrm{d}t)^2 \end{aligned} \quad (1)$$

where $\rho^2 = r^2 + a^2\cos^2\theta\,, \quad \Delta = r^2 - 2Mr + a^2$ and

$$K^a = t^a + \frac{a}{r_H^2 + a^2}\varphi^a$$

Although both $t^a$ and $\phi^a$ are spacelike near $\mathcal{H}$, $K^a K_a \to 0$ as $r \to r_H$

**Carter's Rigidity Theorem:** (Carter 73)

Consider a stationary-axisymmetric black hole with $t^a$ and $\varphi^a$ which satisfy $t \wedge \mathrm{d}t \wedge \varphi = 0 = \varphi \wedge \mathrm{d}\varphi \wedge t$

There exists a Killing field $K^a$ of the form

$$K^a = t^a + \Omega_H \varphi^a$$

which is normal to $\mathcal{H}$. The constant $\Omega_H$ is called the angular velocity of $\mathcal{H}$. Furthermore, the surface gravity $\kappa$ must be constant over $\mathcal{H}$

**Remarks:**

– *purely geometrical result: no use of Einstein's field equations*

– *leaves open the possibility that there could exist stationary BHs whose event horizons are not Killing horizons*

# BH Mechanics and thermodynamics

Mechanics of stationary black holes (Bardeen, Carter & Hawking 73)

$$\kappa = const.\,, \quad \delta M = \frac{1}{8\pi}\kappa\delta A + \Omega_H \delta J$$

$M$: Mass, $\kappa$: Surface gravity, $\Omega_H$: Horizon angular velocity, $J$: Angular momentum

The dominant energy conditions, $T_{ab}V^aW^a \geqslant 0$ for any causal vectors $V^a$, $W^a$

$\Rightarrow$ Mathematical analogue of 0th & 1st laws of equilibrium thermodynamics

$$T = const.\,, \quad \delta E = T\delta S + \text{work term}$$

Quantum effects $\Rightarrow$ $T = \kappa/2\pi$ (Hawking 75)

# Indication of BH uniqueness

If a stationary black hole corresponds to an equilibrium thermodynamic system, then such a stationary BH should be described by merely a small numbers of parameters

# Towards BH uniqueness theorems

**Topology Theorem:** (Hawking 73)

Let $(M, g)$ be a stationary predictable black hole spacetime that satisfies the dominant energy conditions

Spatial cross-section, $\Sigma$, of each connected component of the event horizon $\mathcal{H}$ is homeomorphic to a $2$-sphere

# Towards BH uniqueness theorems

**Rigidity Theorem:** (Hawking 73)

Let $(M, g)$ be an asymptotically flat, regular stationary, predictable black hole spacetime that is a vacuum or electro-vacuum solution to Einstein's equations. Assume further that $(M, g)$ be analytic

The event horizon $\mathcal{H}$ must be a Killing horizon
If rotating, the BH spacetime must be axisymmetric

**Remarks:**

– *makes no assumptions of symmetries beyond stationarity*

– *makes use of Einstein's field equations*

– *use the result of Topology theorem:* $\Sigma \approx S^2$

# Uniqueness theorems

**Uniqueness Theorems:**
(Israel-Carter-Robinson-Mazur-Bunting-Chrusciel)

Let $(M, g)$ be a regular, stationary predictable BH solution of a vacuum or electro-vacuum Einstein's equations. Furthermore, assume $(M, g)$ be analytic and $\mathcal{H}$ be connected

The BH is uniquely specified by its mass, angular momentum, and charges

**Remarks:**

– *vacuum rotating black hole spacetime* $\Rightarrow$ *Kerr-metric*

– *based on the results of Topology and Rigidity theorems*

# Uniqueness theorems

- If weak cosmic censorship (Penrose) holds, gravitational collapse always forms a black hole

  $\Leftarrow$ - *strong support from e.g., BH thermodynamics*

- The Kerr-metric is stable (Press-Teukolsky 73, Whiting 89)

  $\Rightarrow$ - *describes a possible final state of dynamics*

  $\Rightarrow$ - *describes astrophysical black holes–formed via gravitational collapse–in our universe*

- A proof in smooth (not real-analytic) setup (Ionescu-Klainerman 07)

# Summary: Black holes in $4D$ general relativity

## Asymptotically flat stationary BHs in $4$-dimensions

- Exact solutions $\cdots\cdots\cdots\cdot$ e.g., Kerr metric
- Stability $\cdots\cdots\cdots\cdot$ Stable $\Rightarrow$ final state of dynamics
- Topology $\cdots\cdots\cdots\cdot$ Shape of the horizon $\approx 2$-sphere
- Symmetry $\cdots\cdots\cdots\cdot$ Static or axisymmetric
- Uniqueness $\cdots\cdots\cdots\cdot$ Vacuum $\Rightarrow$ Kerr-metric
- BH Thermodynamics $\cdots\cdots\cdots\cdot$ Quantum aspects

**Which properties of $4D$ BHs are extended to $D>4$?**

- **Exact Solutions** — much larger variety

  Rotating Holes (Myers & Perry 86) Rotating Rings (Emparan & Reall 02)

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  Static vacuum $\Rightarrow$ stable (AI & Kodama 03)

  Rotating holes $\Rightarrow$ partial results:

  Special case (e.g., Kunduri-Lucietti-Reall 06, Murata-Soda 07)

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- **Uniqueness** — non-unique Hole and Rings w/ the same $(J, M)$

  Static holes: e.g., (Gibbons, Ida & Shiromizu 02)

  Uniqueness in $5D$ rotating holes/rings (Morisawa-Ida 04, Hollands & Yazadjiev 07)

  (Morisawa, Tomizawa & Yasui 07)

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- BH Thermodynamics — generalize to $D > 4$ e.g., (Rogatko 07)

# Exact solutions in $D > 4$

- Static spherical holes in $\forall D > 4$ (Tangherlini 63)
- Stationary rotating black holes in $\forall D > 4$ (Myers-Perry 82)
  - Topology of horizon cross-sections $\approx S^{D-2}$
  - $[(D+1)/2]$ commuting Killing fields $\Rightarrow$ $[(D-1)/2]$ spins
  - for $D = 4, 5$, $\exists$ Kerr upper-bound on angular momentum $J$
  - for $D \geqslant 6$, No upper-bound on $J$ $\Rightarrow$ ultra-spinning

$$\exists \text{ horizon} \Leftrightarrow 0 = g^{rr} = \Pi_i \left(1 + \frac{(J_i/M)^2}{r^2}\right) - \frac{GM}{r^{D-3}}$$

as the last term dominates for small $r$ when $D \geqslant 6$

Sufficient conditions for no-bound:

– two $J_i = 0$ for $D(odd) \geqslant 7$, one $J_i = 0$ for $D(even) \geqslant 6$

# Exact solutions in $D > 4$

*Stationary, rotating black-rings in* $D = 5$ (Emparan-Reall 02)

- – Topology of the horizon $\approx S^1 \times S^2$
- – $3$-commuting Killing fields $\mathrm{Isom}$: $\mathbb{R} \times SO(2) \times SO(2)$
- – not uniquely specified by global charges $(M,\ J_1,\ J_2)$

  *two* ring-solutions w/ the *same* $(M,\ J_1,\ J_2 = 0)$

$\Rightarrow$ In $D = 5$, Uniqueness Theorem no longer holds as it stands

# Exact solutions in $D > 4$

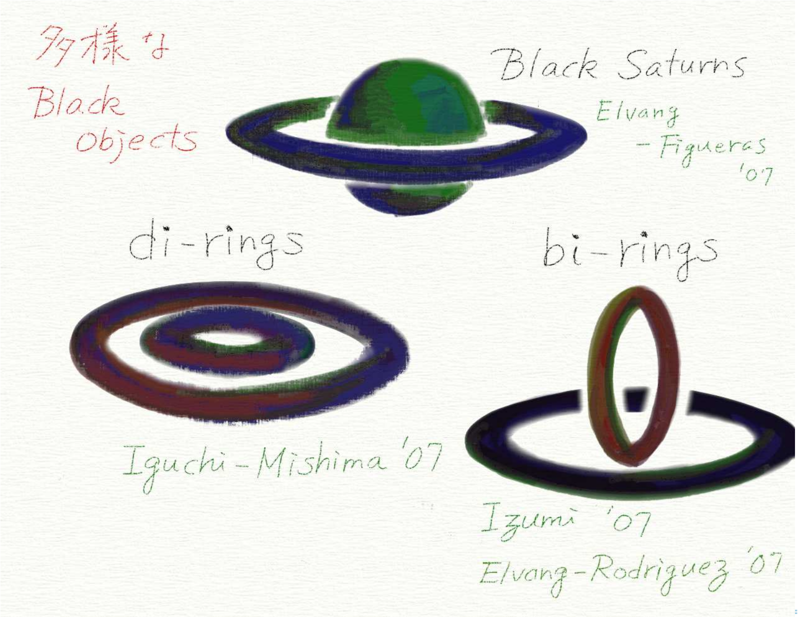
– Solutions akin to Emparan-Reall's ring $(M, J_1 \neq 0, J_2 = 0)$

- Black-ring w/ $(M,\ J_1 = 0,\ J_2 \neq 0)$ (Mishima & Iguchi 05)
- Black-ring w/ *two* angular momenta $(M,\ J_1 \neq 0,\ J_2 \neq 0)$
  (Pomeransky & Sen'kov 06)
  Uniqueness proof (Morisawa-Tomizawa & Yasui 07)

# Exact solutions in $D > 4$

Multi-black vacuum solutions:

- Black di-rings (“ring” $+$ “ring”) (Iguchi & Mishima 07)
- Black-Saturn (“hole” $+$ “ring”) (Elvang & Figueras 07)
- Orthogonal-di-/Bicycling-Rings (“ring” $+$ “ring”)
  (Izumi 07 Elvang & Rodriguez 07)


多様な
Black
objects
Black Saturns
Elvang
-Figueras
'07
di-rings
bi-rings
Iguchi-Mishima '07
Izumi '07
Elvang-Rodriguez '07

# Black holes, rings on various manifolds

- MP in AdS or dS e.g., (Gibbons, Lu, Page, Pope 04)
- – on Gibbons-Hawking e.g. (BMPV 97, Gauntlett-Gutowski-Hull-Pakis-Reall 03)
  – on $4$-Euclid space e.g. (Elvang-Emparan-Mateos-REall 04)
- – on Kaluza-Klein

  – on Eguchi-Hanson

  e.g., (Ishihara, Kimura, Matsuno, Nakagawa, Tomizawa 06-08)

  – on Taub-NUT e.g., (Bena-Kraus-Warner 05)
- Black-p-branes
- Black holes on Braneworld

# Studies of $D > 4$ black holes

- $4D$ Black holes:

  $\Rightarrow$ *"Special" in many respects*

- $D > 4$ Black holes: *Much larger variety*

  $\Rightarrow$ Classify (or get phase space diagram for) them!

  – need study

  - Dynamical stability
  - Possible horizon topology
  - Symmetry properties

# Stability of static black holes

Gravitational perturbations of $\forall D > 4$ static black holes

– $3$ types: *tensor-, vector-, scalar-type* w.r.t. $(D-2)$-base space

- – get a single *master equation* for each type of perturbations (Kodama & AI 03)

  $\Rightarrow$ make complete stability analysis possible

  $\Rightarrow$ Stable for vacuum case (AI & Kodama 03)

- – Einstein-$\Lambda$-Maxwell case: not completed yet

- – New ingredient in $D \geqslant 5$

  Tensor-mode w.r.t. $(D-2)$-horizon manifold $\Sigma$

  **c.f.** if $\Sigma$ is a *highly clumpy* Einstein-manifold,

  $\Rightarrow$ tensor-mode instability (Gibbons & Hartnoll 02)

Static background $D = m + n$ (warped product type) metric:

$$ds^2_{(D)} = {}^{(m)}g_{AB}\mathrm{d}y^A\mathrm{d}y^B + r(y)^2\mathrm{d}\sigma^2_{(n)}$$

e.g., when $m = 2$

$${}^{(2)}g_{AB}\mathrm{d}y^A\mathrm{d}y^B = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2\,, \quad \mathrm{d}\sigma^2_{(D-2)} = n\text{-Einstein}$$

Master equations for each *tensor/vector/scalar*-type of perturbations:

$$\frac{\partial^2}{\partial t^2}\Phi = -A\Phi = \left(\frac{\partial^2}{\partial r_*^2} - U(r)\right)\Phi$$

– looks just like a Schroedinger equation: If $A \geqslant 0 \;\Rightarrow$ stable

# Stability of rotating holes: Some partial analysis

Perturbations on cohomogeneity-1 MP holes:

$D = \text{odd}$, $J_1 = J_2 = \cdots J_{[(D-1)/2]}$
$\Rightarrow$ enhanced symmetry: $\mathbb{R} \times U((D-1)/2) \Rightarrow$ depends only on $r$
$\Rightarrow$ Perturbation equations reduce to *ODEs*

- $D(\text{odd}) \geqslant 7$: $\Rightarrow$ Stable w.r.t. a subclass of tensor perturbations
  (tensor-modes w.r.t. $(D-3)$-base space)
  (Kunduri-Lucietti-Reall 06)

- $D = 5$: Decoupled master equations for zero-modes of vector and tensor (gravity) fields

$\Rightarrow$ Towards complete stability analysis of (cohomo-1) MP holes
(Murata & Soda 07)

# Stability of rotating holes: Some partial analysis

Perturbations on cohomogeneity-2 MP holes:

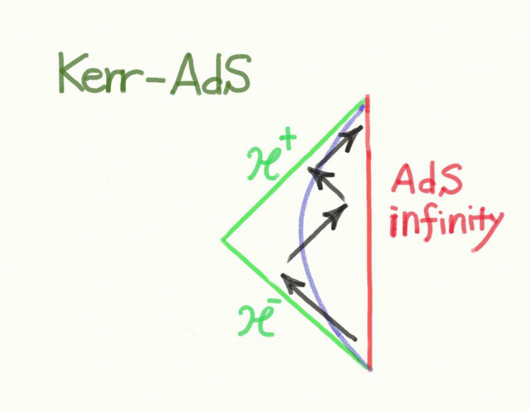
A single rotation: symmetry enhance $U(1)^N \Rightarrow U(1) \times SO(D-3)$

$$ds_D^2 = ds_{(4)}^2 \text{(looks like } D = 4 \text{ Kerr metric)} + r^2 \cos^2\theta \cdot d\Omega_{(D-4)}^2$$

For $D \geqslant 7$, decoupled master equation for tensor-type perturbations w.r.t. $(D-4)$-base space $\Leftrightarrow$ *Massless Klein-Gordon equation*

– Conserved energy integral $\Rightarrow$ Stable
– AdS case $\Rightarrow$ *supperradiant instability* for $|\Lambda| > a^2/r_H^4$ (?)
(Kodama 07)

– observed also in cohomo-1 AdS-MP-holes
(Kunduri-Lucietti-Reall 06)




$$E := -\int_S dSn^a\chi^bT_{ab} \quad \chi^a\text{: co-rotate Killing vector} \tag{2}$$

Note: $\chi^a$ can be non-spacelike if $|\Lambda| \leqslant a^2/r_H^4 \Rightarrow E \geqslant 0$

# Stability of rotating holes: Some partial analysis

Indication of instability of cohomogeneity-2 MP holes
(heuristic argument)

- $D \geqslant 6 \Rightarrow$ no upper-bound on $J$:

  — ultra-spinning hole looks like "pancake"

  $\Rightarrow$ looks like black-p-brane near the rotation axis

  $\Rightarrow$ – expected to be unstable due to Gregory-Laflamme modes (Emparan & Myers 03)

## Myers-Perry solution:

$$
\begin{aligned}
\mathrm{d}s^2 \quad &= \quad -\mathrm{d}t^2 + \frac{M}{\rho^2 r^{D-5}}(\mathrm{d}t + a\sin^2\theta \mathrm{d}\phi)^2 + \frac{\rho^2}{\Delta}\mathrm{d}r^2 \\
&\quad +\rho^2\mathrm{d}\theta^2 + (r^2+a^2)\sin^2\theta\mathrm{d}\phi^2 + r^2\cos^2\theta\mathrm{d}\Omega^2_{(D-4)}
\end{aligned}
$$

where

$$
\rho^2 = r^2 + a^2\cos^2\theta \quad \Delta = r^2 + a^2 - \frac{M}{r^{D-5}}
$$

In the ultra-spinning limit: $a \to \infty$ with $\mu = M/a^2$ kept finite, near the pole $\theta = 0$ ($\sigma := a\sin\theta$) the metric becomes

$$
\mathrm{d}s^2 = -\left(1 - \frac{\mu}{r^{D-5}}\right)\mathrm{d}t^2 + \left(1 - \frac{\mu}{r^{D-5}}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{(D-4)} + \mathrm{d}\sigma^2 + \sigma^2\mathrm{d}\phi^2
$$

$\Rightarrow$ Black-membrane metric $\Rightarrow$ Gregory-Laflamme instability

- – Similar heuristic arguments apply also to *thin black-rings*, other multi-rings, Saturns, etc.

  i.e., – They are expected to suffer from GL-instability

# Topology of event horizon

- Method 1: global analysis (Chrusciel & Wald 94)

  – Combine Topological Censorship and Cobordism of spacelike hypersurface $\mathcal{S}$ with boundaries at horizon and infinity

  Topological Censorship $\Rightarrow$ $\mathcal{S}$ is simply connected
  $\Sigma = \partial\mathcal{S}$ is cobordant to $S^{D-2}$ via $\mathcal{S}$

  In $4D \Rightarrow \partial\mathcal{S}$ must be $S^2$

  – powerful method in $4D$ but turns out to be not so in $D \geqslant 6$
  e.g., (Helfgott-Oz-Yanay 05)

# Topology of event horizon

- Method 2: local analysis (Hawking 72)

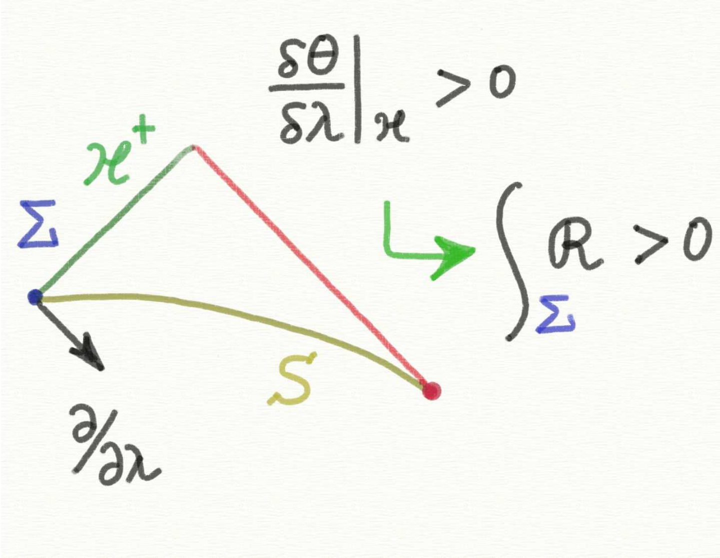
– Combine variational analysis $\delta\theta/\delta\lambda$ and fact that outer-trapped surface must be inside BH, to show

$$\int_\Sigma \mathcal{R} > 0$$

w/ $\Sigma$ being a horizon cross-section and $\mathcal{R}$ scalar curvature of $\Sigma$

$\Rightarrow$ in $4D$, $\Sigma \approx S^2$ via Gauss-Bonnet Theorem

– generalizes to $D > 4$ (Galloway & Schoen 05)


$\frac{\delta\theta}{\delta\lambda}\Big|_{\mathcal{H}} > 0$
$\mathcal{H}^+$
$\Sigma$
$\int_\Sigma R > 0$
$S$
$\partial/\partial\lambda$

**Topology Theorem:** (Galloway & Schoen 05 Galloway 07)

Consider a $\forall D \geqslant 4$ (stationary) black hole spacetime satisfying the dominant energy conditions

The topology of (event) horizon cross-section $\Sigma$ must be such that $\Sigma$ admits metrics of positive scalar curvature

Remarks:

- $\Sigma$ *can be topologically e.g.,* $S^{D-2}$, $S^m \times \cdots \times S^n$
  $\Rightarrow$ *much larger variety in* $D > 4$ *than* $4D$
- $5D \Rightarrow S^3$ *Black holes and* $S^1 \times S^2$ *Black-rings*
- *What if* $\Lambda < 0$ *and* $D \geq 6$? $\Rightarrow$ *more variety?*

# Symmetry property of black holes

**Assertion:**

(1) The event horizon of a stationary, electro-vacuum BH is a Killing horizon

(2) If rotating, the BH spacetime must be axisymmetric

* *Event Horizon*: a boundary of causal past of distant observers

* *Killing Horizon*: a null hypersurface with a Killing symmetry vector field being normal to it

The event horizon is *rigidly* rotating with respect to infinity

$\cdots$ *Black Hole Rigidity*

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$\Rightarrow$ "Rigidity"–if holds also in $D > 4$—places important restrictions on possible exact BH solutions

# Remarks

**An important question:**

- – Does there exist a $D > 4$ BH solution with only two commuting Killing fields (i.e., w/ isom. $\mathbf{R} \times U(1)$ )?
  (Reall 03)

  – all known $D > 4$ BH solutions have multiple rotational symmetries

  $\Rightarrow$ *Hunt (less-symmetric) black objects!*

  – need to show the existence of, *at least, one $U(1)$-symmetry*

# Rigidity theorem in $D = 4$

However Hawking's proof for $4D$ case relies on the fact that event horizon cross-section $\Sigma$ is topologically $2$-sphere

$\Rightarrow$ Generalization to $D > 4$ is not at all obvious

**Goal:** Prove BH Rigidity Theorem in $D \geqslant 4$

No Assumption on Topology of Event Horizon

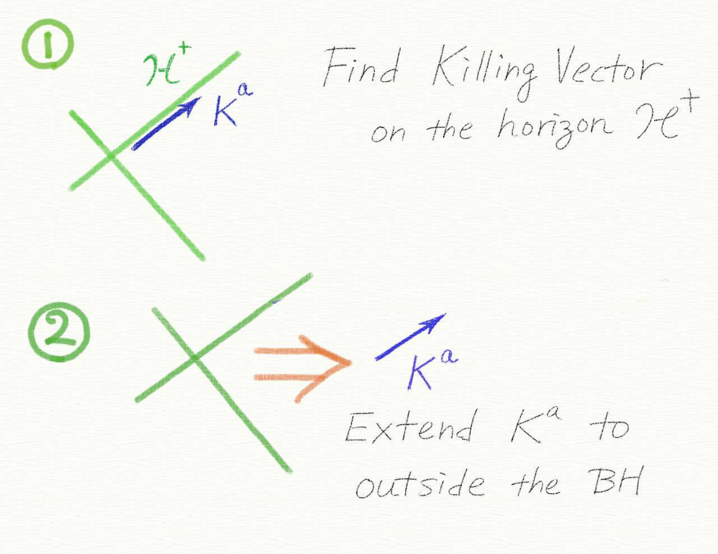
# Rigidity theorems in $D \geqslant 4$ (Hollands, A.I., & Wald 07)

Let $(M, g)$ be a $D \geqslant 4$, analytic, asymptotically flat, stationary vacuum BH solution to Einstein's equation. Assume event horizon $\mathcal{H}$ is analytic, non-degenerate, and topologically $\mathbf{R} \times \Sigma$ with cross-sections $\Sigma$ being compact, connected.

**Theorem 1:** There exits a Killing field $K^a$ in the entire exterior of the BH such that $K^a$ is normal to $\mathcal{H}$ and commutes with the stationary Killing vector filed $t^a$ $\Rightarrow$ "Killing horizon"

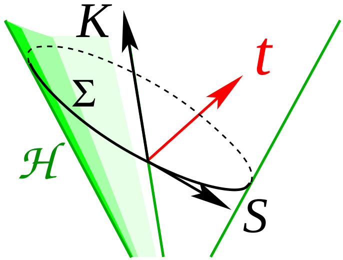
**Theorem 2:** If $t^a$ is not normal to $\mathcal{H}$, i.e., $t^a \neq K^a$, then there exist mutually commuting Killing vector fields $\varphi^a_{(1)}, \cdots, \varphi_{(j)}$ $(j \geq 1)$ with period $2\pi$ and $t^a = K^a + \Omega_{(1)}\varphi^a_{(1)} + \cdots + \Omega_{(j)}\varphi^a_{(j)}$, where $\Omega_{(j)}$'s constants. $\Rightarrow$ "Axisymmetry"

# Brief sketch of proof of Theorem 1



# Sketch of proof of Theorem 1

"Trial foliation" $\Sigma$ &
"candidate" vector $K^a$




$K^a$ depends on $\Sigma$

**Step 1**

Construct a "candidate" Killing field $K^a$ on $\mathcal{H}$ which satisfies

- $K^a K_a = 0$ and $\pounds_t K^a = 0$ on $\mathcal{H}$
- $\pounds_K g_{ab} = 0$ (Killing eqn.) on $\mathcal{H}$
- $\alpha = const.$ ($K^c \nabla_c K^a = \alpha K^a$) on $\mathcal{H}$

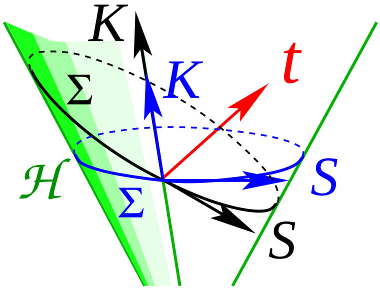
Try this one ! $K^a = t^a - s^a$

**Step 2**

- Show Taylor expansion $\partial^m (\pounds_K g_{ab}) / \partial \lambda^m = 0$ at $\mathcal{H}$
- Extend $K^a$ to the entire spacetime by invoking analyticity

However, there is No reason why $\alpha$ need be constant

— wish to find "correct" $\tilde{K}^a$ with $\tilde{\alpha} = const. =: \kappa$ on $\mathcal{H}$ by choosing a new "correct" foliation $\tilde{\Sigma}$




$$K^a + s^a = t^a = \tilde{K}^a + \tilde{s}^a$$

Both $K^a$ and $\tilde{K}^a$ are null

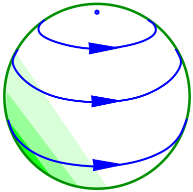
$$\tilde{K}^a = f(x)\, K^a$$

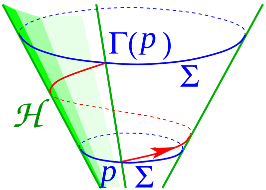
**Task:** Find a solution to equation for coordinate transformation from trial $\Sigma$ to correct $\tilde{\Sigma}$:

$$-\pounds_s\, f(x) + \alpha(x)\, f(x) = \kappa$$

When one solves this equation, the spacetime dimensionality comes to play a role

# Find correct foliation $\tilde{\Sigma}$: $4D$ case Hawking 73




In $4D$, horizon cross-section $\Sigma$ is $2$-shere, and therefore the orbits of $s^a$ must be closed

There is a discrete isometry "$\Gamma$" which maps each null generator into itself

Discrete isometry, $\Gamma$, helps to

- — define the surface gravity as $\kappa \equiv P^{-1} \int_0^P \alpha[\phi_s(x)]ds$
- — find correct foliation $\tilde{\Sigma}$
- — show **Step 2**

# Find correct foliation $\tilde{\Sigma}$: $D > 4$ case

No reason that the isometry $s^a$ need have closed orbits on $\Sigma$.
$\Rightarrow$ in general, there is No discrete isometry $\Gamma$.

e.g., $5D$ Myers-Perry BH w/ $2$-rotations $\Omega_{(1)}$, $\Omega_{(2)}$:

$$\Sigma \approx S^3 \,, \quad t^a = K^a + s^a$$

$$s^a = \Omega_{(1)} \varphi^a_{(1)} + \Omega_{(2)} \varphi^a_{(2)}$$

Each rotation Killing vector $\varphi^a$ has closed orbits
but $s^a$ does not if $\Omega_{(1)}$ and $\Omega_{(2)}$ are incommensurable

# Solution to $D > 4$ case:

**(i)** When $s^a$ has closed orbits on $\Sigma \Rightarrow$ we are done!

$$\kappa = \frac{1}{P}\int_0^P \alpha[\phi_s(x)]\mathrm{d}s \quad P : \text{period} \quad \phi_s : \text{isom. on } \Sigma \text{ by } s^a$$

**(ii)** When $s^a$ has No closed orbits $\Rightarrow$ Use Ergodic Theorem !

$$\kappa = \lim_{T\to\infty}\frac{1}{T}\int_0^T \alpha[\phi_s(x)]\mathrm{d}s = \frac{1}{\mathrm{Area}(\Sigma)}\int_\Sigma \alpha(x)\mathrm{d}\Sigma$$

"time-average" "space-average"

- — can show that the limit "$\kappa$" exists and is constant
- — can find well-behaved transformation $\Sigma \to \tilde{\Sigma}$

# Solution to $D > 4$ case:

— wish to solve equation, $\alpha(x)f(x) - \pounds_s f(x) = \kappa$
to find the "correct" horizon Killing field, $\tilde{K}^a = f(x)\, K^a$

**Solution:**

$$f(x) = \kappa \int_0^\infty P(x,T)\mathrm{d}T\,, \quad P(x,T) = \exp\left(-\int^T \alpha[\phi_s(x)]\mathrm{d}s\right)$$

— since $\forall\epsilon > 0$, $P(x,T) < e^{(\epsilon-\kappa)T}$, for sufficiently large $T$,
$f(x)$ above is well-defined

# Brief sketch of proof of Theorem 2

— wish to show $t^a = K^a + \Omega_{(1)}\varphi^a_{(1)} + \cdots + \Omega_{(j)}\varphi^a_{(j)}$

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- — Extend $U(1)^N$ into the entire spacetime by analyticity

# Remarks

**Immediate generalizations:**

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**Immediate generalizations:**

- — can apply to Einstein-$\Lambda$-Maxwell system
  e.g., charged-AdS-BHs

- — combined together with Staticity Theorems

  $d = 4$ Sudarsky & Wald (92) $d > 4$ Rogatko (05)

$\Rightarrow$ The assertion is rephrased as

> Stationary, non-extremal BHs in $D \geqslant 4$ Einstein-Maxwell system are either static or axisymmetric

# Remarks

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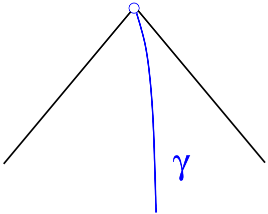
  e.g., cosmological horizon

Cosmological horizon



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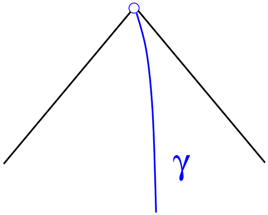
  e.g., cosmological horizon

- — can remove analyticity assumption for the BH interior

  by using initial value formulation w/ initial data for $K^a$ on the bifurcate horizon

Cosmological horizon



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  $\Rightarrow$ Horizon Killing field $K^a$ may not have a single-valued analytic extension

- Extremal BHs (i.e., BHs w/ degenerate horizon $\kappa = 0$)