$\mathcal{N} = 2$ Liouville theory and its application to NS5 brane

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and

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1 Introduction

1.1 What is Liouville Filed Theory?

- LFT appears in the quantization of Noncritical string (Polyakov, DDK ...)
- Noncompact and Irrational CFT (← difficult to solve)

Action

$$S = \frac{1}{4\pi}\int d^2z\sqrt{g}(g^{ab}\partial_a\phi\partial_b\phi + QR\phi + 4\pi\mu e^{2b\phi})$$

$$Q = b + b^{-1}, \qquad c = 1 + 6Q^2$$

10 years ago (matrix revolution)
(BK,DS,GM...)it was found that

it is equivalent to Double Scaling Matrix Model .

→ LFT is exactly solvable?

1.2 10 years have past since then...

1. Developments of LFT itself

- 3pt function (DOZZ formula)
- brane (FZZT/ZZ brane)

2. Matrix Model reloaded ($\sim$ 2003)

- Matrix is tachyon field on D-brane
- Based on the holographic principle
- A kind of gauge/gravity correspondence
- Can be understood from VSFT
- Conncection to topological string theory...

See *e.g.* my review paper (hep-th/0402009).

However, bosonic (or $\mathcal{N} = 1$) Liouville theory has $c = 1$ barrier. So it is not so clear how we can apply the results to higher dimensional string theory.

1.3 $\mathcal{N}=2$ Liouville theory

In this talk, we consider $\mathcal{N}=2$ Liouville theory.

$\mathcal{N}=2$ Liouville theory is special because it has no $\hat{c}=1$ barrier.

Action

$$S=\frac{1}{2\pi}\int d^2z d^4\theta S S^\dagger+\mu\left(\int d^2z d^2\theta e^{\frac{1}{\mathcal{Q}}S}+h.c.\right)$$

Central charge $\hat{c}=\frac{c}{3}=1+\mathcal{Q}^2$ ($\mathcal{Q}$ is background charge: $\int d^2z\mathcal{Q}RS$) so there is no $\hat{c}=1$ barrier.

Application of $\mathcal{N}=2$ Liouville theory

- 2-dimensional black hole
- World sheet description of NS5-brane
- Singular CY spaces
- Branes/Orientifolds in these backgrounds
- Time-dependent physics (rolling D-brane)

Plan of my talk

1. Introduction
2. $\mathcal{N} = 2$ Liouville theory
   - Bulk theory (duality of $\mathcal{N} = 2$ LFT)
   - Target space geometry
   - Boundary states
   - Crosscap states
3. Application to NS5 background
   - Stable non-BPS brane (hairpin brane)
   - Rolling D-brane
   - Emission rate from rolling D-brane

2 $\mathcal{N}=2$ Liouville theory

2.1 Bulk theory

Action of $\mathcal{N}=2$ LFT

Action

$$S=\frac{1}{2\pi}\int d^2zd^4\theta SS^\dagger$$

S : chiral super field $(\bar{D}_\pm S=0)$

$S=\Phi+iY$ with background charge for $\Phi:\mathcal{Q}$

Interaction term preserving $\mathcal{N}=2$ SCFT

1. Super potential (F-term): $\int d^2zd^2\theta e^{\frac{1}{\mathcal{Q}}S}+h.c.$

$$S_++S_-=\int d^2z\frac{\mu}{\mathcal{Q}^2}\psi^+\psi^-e^{\frac{1}{\mathcal{Q}}S}+\frac{\mu}{\mathcal{Q}^2}\bar{\psi}^+\bar{\psi}^-e^{\frac{1}{\mathcal{Q}}S^*}$$
$$+\frac{\pi\mu^2}{\mathcal{Q}^2}:e^{\frac{1}{\mathcal{Q}}S}::e^{\frac{1}{\mathcal{Q}}S^*}:$$

2. Kähler potential (D-term): $\int d^2zd^4\theta e^{\frac{\mathcal{Q}}{2}(S+S^\dagger)}$

$$S_{nc}=\tilde{\mu}\int d^2z(\partial\phi-i\partial Y-\mathcal{Q}\psi^+\bar{\psi}^+)$$
$$\times(\bar{\partial}\phi+i\bar{\partial}Y-\mathcal{Q}\psi^-\bar{\psi}^-)e^{\mathcal{Q}\phi}$$

Central charge: $\hat{c}=\frac{c}{3}=1+\mathcal{Q}^2$.

2.2 Duality of $\mathcal{N} = 2$ LFT

$\mathcal{N} = 2$ LFT with superpotential (1) is equivalent to the LFT with Kähler potential (2).

(Conjectured in hep-th/0210208: Ahn et. al.)

I have given some supporting arguments in hep-th/0402009.

- $\mathcal{N} = 2$ Liouville theory (with F-term) is mirror dual to $SL(2,\mathrm{R})/U(1)$ supercoset model (Hori-Kapustin)
- Wakimoto screening charge of the $SL(2,R)/U(1)$ is nothing but D-term (upto BRST exact term).
- From the space-time perspective, F-term is complex structure deformation, and D-term is Kähler deformation.
- In case of K3, this is just a convention. In general they are mirror to each other.

2.3 Target space geometry of $\mathcal{N}=2$ LFT 1

Typically, $\mathcal{N} = 2$ LFT represents the noncompact direction of the singular CY space.

Example 1

NS5-brane solution:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \left(1 + \frac{N\alpha'}{r^2}\right) dx^m dx^m,$$
$$e^{2(\Phi-\Phi_0)} = 1 + \frac{N\alpha'}{r^2}, \qquad H_{mnp} = -\epsilon^q_{mnp}\partial_q \Phi$$

In CHS limit (near horizon limit) we can introduce linear dilaton coordinate

$$\phi = \sqrt{N\alpha'} \ln \sqrt{\frac{r^2}{N\alpha'}},$$

The target space becomes

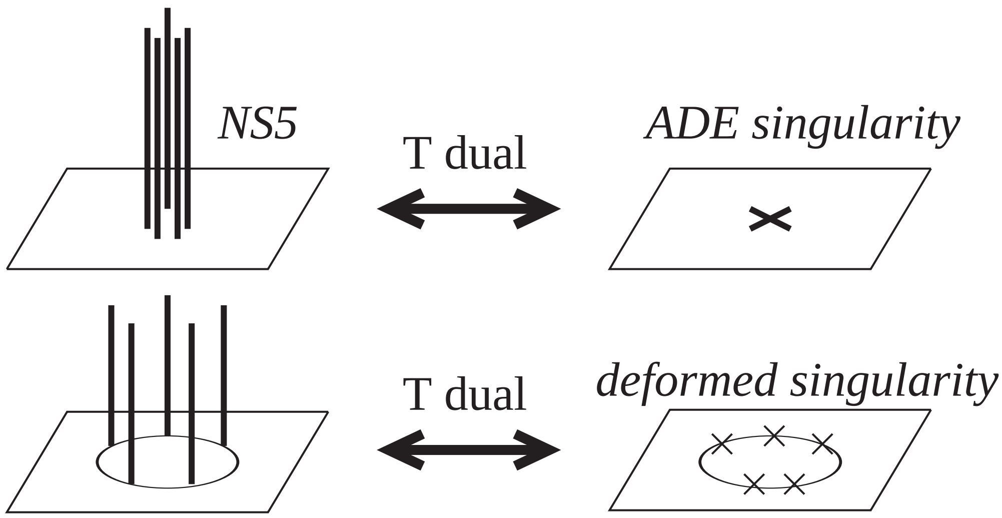
$$\mathrm{R}^{5,1} \times \mathrm{R}_\phi \times SU(2)_{N-2} \cong \mathrm{R}^{5,1} \times \frac{\left[\mathrm{R}_\phi \times S^1_Y\right] \times M_{N-2}}{\mathrm{Z}_N}$$

M_k is level k $\mathcal{N}=2$ minimal model and $\mathrm{R}_\phi \times S^1_Y$ is the $\mathcal{N}=2$ LFT ($\mathcal{Q} = \sqrt{2/N}$)! $\rightarrow$ Solvable!

2.4 Target space geometry of $\mathcal{N}=2$ LFT 2

Like CHS background we can embed $\mathcal{N}=2$ LFT into Superstring.

→ radial direction of NS5 or singular CY



- Deformation by Super potential is resolution of singularity from complex moduli.

$$z_1^N + z_2^2 + z_3^2 = 0 \rightarrow z_1^N + z_2^2 + z_3^2 = \mu$$

- Deformation by Kähler potential is resolution of singularity from Käler moduli.

We can study these singular CY spaces by studying $\mathcal{N}=2$ LFT

2.5 Branes/Orientifold in $\mathcal{N} = 2$ LFT

We will construct boundary/crosscap states in $\mathcal{N} = 2$ LFT from the modular bootstrap method.

Remark 1

- Boundary states were first constructed by Eguchi-Sugawara in hep-th/0311141.
- I constructed crosscap states in hep-th/0409039.
- Modular bootstrap is consistent with conformal bootstrap method (Ahn et.al, Hosomichi...).
- The modular bootstrap method was first introduced in the context of bosonic ZZ brane (→ now becomes almost standard to solve non-compact BCFT).

2.6 Modular bootstrap 1

Philosophy of modular bootstrap method

> We assume the spectrum of the open string and solve the Cardy condition from the ansatz.

It goes schematically as follows ($T = -1/t$):

1. We first assume the existence of identity brane:

$$\langle B, O|e^{-TH}|B, O\rangle = \chi_O(it) = \mathrm{Tr}_{open,O}e^{2\pi itH}$$

2. Then we can solve boundary states $|B, O\rangle$.

3. Another overlaps are given by the general characters:

$$\langle B, k|e^{-TH}|B, O\rangle = \chi_k(it) = \mathrm{Tr}_{open,k}e^{2\pi itH}$$

4. We have various boundary states $|B, k\rangle$ represented by the open string character k.

5. Finally, we should check the Cardy condition by studying overlaps between boundary states:

$$\langle B, k|e^{-TH}|B, k'\rangle \stackrel{?}{=} \sum_{f(k,k')} \chi_{f(k,k')}(it)$$

2.7 Modular bootstrap 2

To complete the procedure, we need to know the modular transformation of the $\mathcal{N}=2$ character.

For simplicity, we consider the decompactified version of $\mathcal{N}=2$ LFT.

We set $q=e^{2\pi i\tau}$ and $y=e^{2\pi iz}$ as usual.

(1) character of massive representation

$$\mathrm{ch}^{(NS)}(h,Q;\tau,z)=q^{h-\frac{\hat{c}-1}{8}}y^{Q}\frac{\theta_3(\tau,z)}{\eta(\tau)^3}\ .$$

(2) character of massless representation

$$\mathrm{ch}_M^{(NS)}(Q;\tau,z)=q^{\frac{|Q|}{2}-\frac{\hat{c}-1}{8}}y^{Q}\frac{1}{1+y^{\mathrm{sgn}(Q)}q^{\frac{1}{2}}}\frac{\theta_3(\tau,z)}{\eta(\tau)^3}\ .$$

(3) character of graviton (identity) representation

$$\mathrm{ch}_G^{(NS)}(\tau,z)=q^{-\frac{\hat{c}-1}{8}}\frac{1-q}{(1+yq^{\frac{1}{2}})(1+y^{-1}q^{\frac{1}{2}})}\frac{\theta_3(\tau,z)}{\eta(\tau)^3}\ .$$

2.8 Modular Transformation

Modular transformation of massive character:

$$\mathrm{ch}^{(NS)}(p,\omega;-\frac{1}{\tau},\frac{z}{\tau}) = e^{i\pi\frac{\hat{c}z^2}{\tau}}\frac{2}{\mathcal{Q}}\int_0^\infty dp'\int_{-\infty}^\infty d\omega' \\ e^{-2\pi i\frac{\omega\omega'}{\mathcal{Q}^2}}\cos(2\pi pp')\mathrm{ch}^{(NS)}(p',\omega';\tau,z)$$

Modular transformation of identity character:

$$\mathrm{ch}_G^{(NS)}(-\frac{1}{\tau},\frac{z}{\tau}) = e^{i\pi\frac{\hat{c}z^2}{\tau}}\frac{1}{\mathcal{Q}}\int_0^\infty dp'\int_{-\infty}^\infty d\omega' \\ \frac{\sinh(\pi\mathcal{Q}p)\sinh(2\pi\frac{p}{\mathcal{Q}})}{|\cosh\pi(\frac{p}{\mathcal{Q}}+i\frac{\omega}{\mathcal{Q}^2})|^2}\mathrm{ch}^{(NS)}(p',\omega';\tau,z) \\ +\text{discrete terms}$$

Massless care is obtained similarly.

Remark 2

Discrete terms are necessary and contain topological information.

2.9 Boundary states

We expand Cardy boundary state as

$$\langle B, O| = \frac{2}{\mathcal{Q}} \int_0^\infty dp \int_{-\infty}^\infty d\omega \Psi_O(p,\omega)_B\langle\langle p,\omega| \ .$$

with the normalization

$$\begin{aligned} & {}_B\langle\langle p,\omega| e^{-\pi T H^{(c)}} |p',\omega'\rangle\rangle_B \\ &= \delta(p-p')\mathcal{Q}\delta(\omega-\omega')\mathrm{ch}^{(NS)}(p,\omega;iT) \ , \end{aligned}$$

Solving

$$\langle B, O| e^{-TH} |B, O\rangle = \mathrm{ch}_G^{(NS)}(it)$$

by taking the "square root" of modular transformation, we obtain (Class 1 brane)

$$\Psi_O(p,\omega) = \frac{1}{\mathcal{Q}} \frac{\Gamma(\frac{1}{2}+\frac{\omega}{\mathcal{Q}^2}-i\frac{p}{\mathcal{Q}})\Gamma(\frac{1}{2}-\frac{\omega}{\mathcal{Q}^2}-i\frac{p}{\mathcal{Q}})}{\Gamma(-i\mathcal{Q}p)\Gamma(1-i\frac{2p}{\mathcal{Q}})}$$

Then

$$\langle B, p', \omega'| e^{-TH} |B, O\rangle = \mathrm{ch}^{(NS)}(p',\omega';it)$$

implies the Class 2 (noncompact) brane

$$\begin{aligned} \Psi_{p',\omega'}(p,\omega) &= \mathcal{Q} e^{2\pi i \frac{\omega\omega'}{\mathcal{Q}^2}} \cos(2\pi p p') \\ & \frac{\Gamma(i\mathcal{Q}p)\Gamma(1+i\frac{2p}{\mathcal{Q}})}{\Gamma(\frac{1}{2}+\frac{\omega}{\mathcal{Q}^2}+i\frac{p}{\mathcal{Q}})\Gamma(\frac{1}{2}-\frac{\omega}{\mathcal{Q}^2}+i\frac{p}{\mathcal{Q}})} \end{aligned}$$

2.10 Modular bootstrap for crosscap states

We can also construct the crosscap state from the modular bootstrap method.

The modular bootstrap assumption is ($t = -1/4T$)

$$\langle C-|e^{-\pi T H^{(c)}}|B,O,+\rangle = \mathrm{ch}^{(NS)}_{G,\Omega}(it) = \mathrm{Tr}_{open,O}\Omega e^{2\pi i t H},$$

where Ω inserted character (Möbius strip) is

$$\mathrm{ch}^{(NS)}_{G,\Omega}(\tau) = q^{-\frac{\hat{c}-1}{8}} \frac{1+q}{(1+iq^{\frac{1}{2}})^2} e^{\frac{\pi i}{8}} \frac{\theta_3(\tau+\frac{1}{2})}{\eta(\tau+\frac{1}{2})^3} \, .$$

The relevant modular transformation ($t = -1/4T$: P = TSTTS) is given by

$$\mathrm{ch}^{(NS)}_{G,\Omega}(-\frac{1}{4\tau}) = \frac{i}{2\mathcal{Q}} \int_{\infty}^{\infty} d\omega' \int_{0}^{\infty} dp'$$

$$\left[\left\{ e^{-\frac{\pi \mathcal{Q} p'}{2}} \frac{1}{1+ie^{\pi(\frac{p'}{\mathcal{Q}}+i\frac{\omega'}{\mathcal{Q}^2})}} + e^{\frac{\pi \mathcal{Q} p'}{2}} \frac{1}{1+ie^{\pi(-\frac{p'}{\mathcal{Q}}-i\frac{\omega'}{\mathcal{Q}^2})}} \right.\right.$$

$$+ e^{-\frac{\pi \mathcal{Q} p'}{2}} \frac{1}{1+ie^{\pi(\frac{p'}{\mathcal{Q}}-i\frac{\omega'}{\mathcal{Q}^2})}} + e^{\frac{\pi \mathcal{Q} p'}{2}} \frac{1}{1+ie^{\pi(-\frac{p'}{\mathcal{Q}}+i\frac{\omega'}{\mathcal{Q}^2})}}$$

$$\left. -e^{-\frac{\pi \mathcal{Q} p'}{2}} - e^{\frac{\pi \mathcal{Q} p'}{2}} \right\}$$

$$\left. \times \, \mathrm{ch}^{(NS)}_{\tilde{\Omega}}(p',\omega';\tau) \right] + \text{discrete terms} \, .$$

2.11 Crosscap states

We expand crosscap state as

$$\langle C-| = \frac{2}{\mathcal{Q}}\int_0^\infty dp \int_{-\infty}^{\infty} d\omega \Psi_c(p,\omega)_C\langle\langle p,\omega,-| \ .$$

Ishibashi crosscap states are defined by

$$\begin{aligned} & {}_B\langle\langle p,\omega,\pm|e^{-\pi T H^{(c)}}|p',\omega',\mp\rangle\rangle_C \\ &= \delta(p-p')\mathcal{Q}\delta(\omega-\omega')\mathrm{ch}_\Omega^{(NS)}(p,\omega;iT) \ . \end{aligned}$$

The modular bootstrap equation

$$\langle C-|e^{-\pi T H^{(c)}}|B,O,+\rangle = \mathrm{ch}_{G,\Omega}^{(NS)}(it) \ ,$$

is solved by

$$\Psi_c(p,\omega) = \Psi_O(-p,\omega)^{-1}P(p,\omega) \ ,$$

where modular transformation matrix $P(p,\omega)$ is given by

$$P(p,\omega) = \frac{i\cosh(\frac{\pi p}{Q})\left[-i\cosh(\frac{\pi \mathcal{Q}p}{2})\cos(\frac{\pi\omega}{\mathcal{Q}^2}) + \sinh(\frac{\pi p}{\mathcal{Q}})\sinh(\frac{\pi Q p}{2})\right]}{\cosh(\frac{2\pi p}{\mathcal{Q}}) + \cos(\frac{2\pi\omega}{\mathcal{Q}^2})}$$

2.12 Consistency checks

The boundary/crosscap states constructed above survives some consistency checks.

- Overlaps between various boundary states satisfy expected Cardy condition.
- Overlaps between compact boundary states and crosscap states satisfy expected Cardy condition.
- However overlaps between noncompact boundary states and crosscap states leave some subtlety (related to IR divergence?).
- Semiclassical limit is consistent with minisuperspace approximation.
- The boundary state is consistent with the conformal bootstrap.

2.13 On the discrete terms

Discrete terms contain topological information

Discrete terms coupling to R-R ground states (chiral primary) can be seen as period integrals:

$$\Pi_i^\gamma = \langle i|B_\gamma\rangle \ , \quad \Pi_i^\Omega = \langle i|C\rangle$$

In the Landau-Ginzburg approach (with nontrivial dilaton),

$$\Pi_i^\gamma = \int dS_\gamma \Phi_i(S) \exp(-\frac{S\mathcal{Q}}{2} - \mu e^{\frac{1}{\mathcal{Q}}S}) \ .$$

$\frac{1}{g_s} = e^{-\frac{S\mathcal{Q}}{2}}$ **is due to linear dilaton. Interestingly, if we make change of variables as** $X = e^{-\frac{S\mathcal{Q}}{2}}$**, we can eliminate the linear dilaton term:**

$$\Pi_i^\gamma = \int dX \Phi_i'(X) \exp(-\mu X^{-\frac{2}{\mathcal{Q}^2}}) \ .$$

This is consistent with the conjecture that LG model with $\mathcal{N} = 2$ LFT is equivalent to LG model with negative power potential $\mu X^{-\frac{2}{\mathcal{Q}^2}}$.

2.14 Applications

Several applications of the boundary/crosscap states of $\mathcal{N} = 2$ Liouville theory include:

- Branes/orientifolds in NS5-brane background
- Supersymmetric matrix models in two dimension
- Branes in two-dimensional black holes
- (Special) Lagrangian geometry of singular CY spaces (intersection number, vanishing cycle...)
- (After Wick rotation) inhomogeneous decay of rolling D-brane (orientifold?) in NS5-brane background (We'll see next)
- Sen's conserving charge (work in progress)

Summary of this section

- $\mathcal{N} = 2$ LFT has a duality property.
  (hep-th/0402009)
- Target space of $\mathcal{N} = 2$ LFT is a singular CY space.
- Boundary states can be constructed from modular bootstrap.
- Crosscap states can be constructed too.
  (hep-th/0409039)

3 Application to NS5-brane background

3.1 CHS revisited

$$\mathrm{R}^{5,1} \times \mathrm{R}_\phi \times SU(2)_{N-2} \ .$$

In the following, we consider the combination $R^1 \times R_\phi$ to make $\mathcal{N} = 2$ LFT.

> This combination leads to decompactified version of $\mathcal{N} = 2$ LFT.

Especially, if we take R^1 as a time-direction, the time dependent physics will show up.

Question 1

What happens to the branes so far constructed in this setup?

> Class 2 brane in the noncompact theory becomes stable non-BPS brane (hairpin brane). It becomes infalling brane into the NS5 core after the Wick rotation

3.2 Non-BPS hairpin brane

Class 2 brane in the noncompact theory becomes stable non-BPS brane (hairpin brane).

$$\Psi(p,q) = \frac{\Gamma(i\mathcal{Q}p)\Gamma(1+i\frac{2p}{\mathcal{Q}})}{\Gamma(\frac{1}{2}+\frac{q}{\mathcal{Q}}+i\frac{p}{\mathcal{Q}})\Gamma(\frac{1}{2}-\frac{q}{\mathcal{Q}}+i\frac{p}{\mathcal{Q}})}$$

In the large N ($\mathcal{Q} = \sqrt{2/N} \to 0$) limit, this becomes the hairpin curve

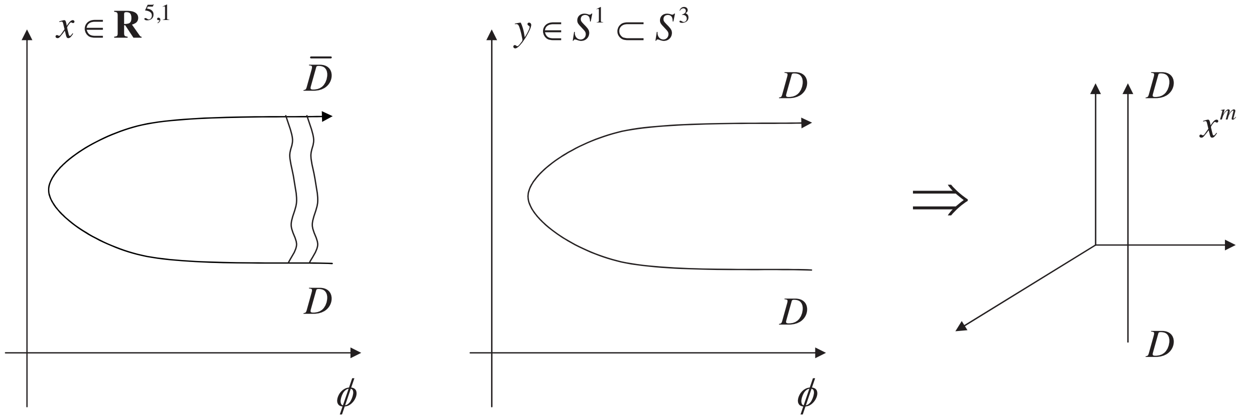
$$e^{-\frac{\mathcal{Q}\phi}{2}} = 2\cos\frac{\mathcal{Q}x}{2} \; .$$

Indeed the Fourier transform of this wavefunction is

$$\Psi(\phi,x) \to \delta(e^{-\frac{\mathcal{Q}\phi}{2}} - 2\cos\frac{\mathcal{Q}x}{2})$$

Exact calculation gives

$$\Psi(\phi,x) \sim \frac{1}{(2\cos\frac{\mathcal{Q}x}{2})^{\frac{2}{\mathcal{Q}^2}+1}} \exp\left[-\frac{\phi}{\mathcal{Q}} - \frac{e^{-\frac{\phi}{\mathcal{Q}}}}{(2\cos\frac{\mathcal{Q}x}{2})^{\frac{2}{\mathcal{Q}^2}}}\right]$$



The direct calculation of $\widetilde{\text{NS}}$ and R sector shows this is non-BPS (in contrast to the compact case).

Remark 3

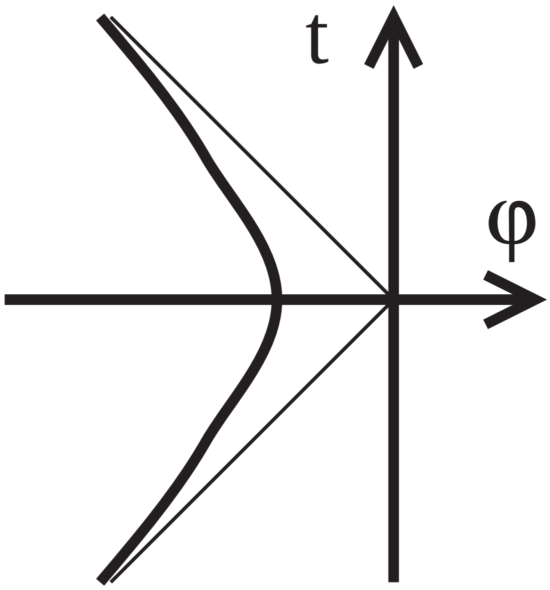
- **This brane is asymptotically D-D̄ pair.**
- **Would be tachyon is massive because of the asymptotic separation**
- **Usual class 2 brane is just a straight line and BPS.**
- **Class 1 brane is wrapping vanishing cycle and W-boson in LST (BPS).**
- **Bosonic hairpin was first introduced by Ribault and Schomerus (hep-th/0310024).**

3.3 Rolling D-brane in NS5 background

The DBI action in the NS5-brane background predicts a rolling D-brane solution (Kutasov).

The classical orbit is given by

$$e^{-\frac{\mathcal{Q}\phi}{2}} = 2\cosh\frac{\mathcal{Q}t}{2}$$



This is nothing but the Wick rotation of the Class 2 (hairpin) D-brane

$$e^{-\frac{\mathcal{Q}\phi}{2}} = 2\cos\frac{\mathcal{Q}x}{2} .$$

In our paper (hep-th/0406173) we constructed this boundary state.

Remark 4

- Kutasov argued that the potential of the rolling D-brane can be mapped to the potential of the unstable D-brane in the flat space.

$$L_T = V(T)\sqrt{1+\partial_\mu T \partial^\mu T}$$
$$L_{DBI} = \frac{1}{\sqrt{H(R)}}\sqrt{1+H(R)\partial_\mu R \partial^\mu R}$$

- We will see that the emission rate is consistent with this correspondence.

- We can also calculate the energy-momentum tensor of the rolling brane from boundary state.

- Energy is related to the bulk cosmological constant ($\longrightarrow$ just the shift of the radial direction)

3.4 Naive Wick rotation

Naive Wick rotation in the momentum space does not work.

Try naive Wick rotation $q \to i\omega$.

$$\Psi^{(NS)}_{naive}(p,\omega) = \frac{\Gamma(iQp)\Gamma(1+i\frac{2p}{Q})}{\Gamma(\frac{1}{2}-i\frac{\omega}{Q}+i\frac{p}{Q})\Gamma(\frac{1}{2}+i\frac{\omega}{Q}+i\frac{p}{Q})} \, .$$

This does not make sense! For example, the absolute square is

$$|\Psi^{(NS)}_{naive}(p,\omega)|^2 \sim \frac{\cosh(\frac{2\pi\omega}{Q})+\cosh(\frac{2\pi p}{Q})}{\sinh(\pi Q p)\sinh(\frac{2\pi p}{Q})} \, ,$$

which is divergent under $\omega \to \infty$.

$\longrightarrow$ no meaningful open string spectrum.

Besides the position space wavefunction (in the classical limit) is not the curve

$$e^{-\frac{Q\phi}{2}} = 2\cosh\frac{Qt}{2}$$

3.5 Correct Wick rotation 1

We should perform Wick rotation in the position space.

$$\tilde{\Psi}^{(NS)}(\phi, t)$$

$$= \left(2\cosh\frac{\mathcal{Q}t}{2}\right)^{\frac{2}{\mathcal{Q}^2}+1} \exp\left[-\frac{\phi}{\mathcal{Q}} - \frac{e^{-\frac{\phi}{\mathcal{Q}}}}{\left(2\cosh\frac{\mathcal{Q}t}{2}\right)^{\frac{2}{\mathcal{Q}^2}}}\right]$$

After the Fourier transformation:

$$\Psi^{(NS)}(p, \omega) = \frac{-i\sqrt{2}\mathcal{Q}\sinh(\frac{2\pi p}{\mathcal{Q}})}{2\cosh[\frac{\pi}{\mathcal{Q}}(p+\omega)]\cosh[\frac{\pi}{\mathcal{Q}}(p-\omega)]} \cdot \frac{\Gamma(i\mathcal{Q}p)\Gamma(1+i\frac{2p}{\mathcal{Q}})}{\Gamma(\frac{1}{2}+i\frac{\omega}{\mathcal{Q}}+i\frac{p}{\mathcal{Q}})\Gamma(\frac{1}{2}+i\frac{\omega}{\mathcal{Q}}+i\frac{p}{\mathcal{Q}})}.$$

We have a nontrivial dumping factor!

Its absolute square behaves nicely:

$$|\Psi^{(NS)}(p, \omega)|^2 = \frac{2\sinh\left(\frac{2\pi p}{\mathcal{Q}}\right)}{\left[\cosh\left(\frac{2\pi\omega}{\mathcal{Q}}\right) + \cosh\left(\frac{2\pi p}{\mathcal{Q}}\right)\right]\sinh(\pi\mathcal{Q}p)}.$$

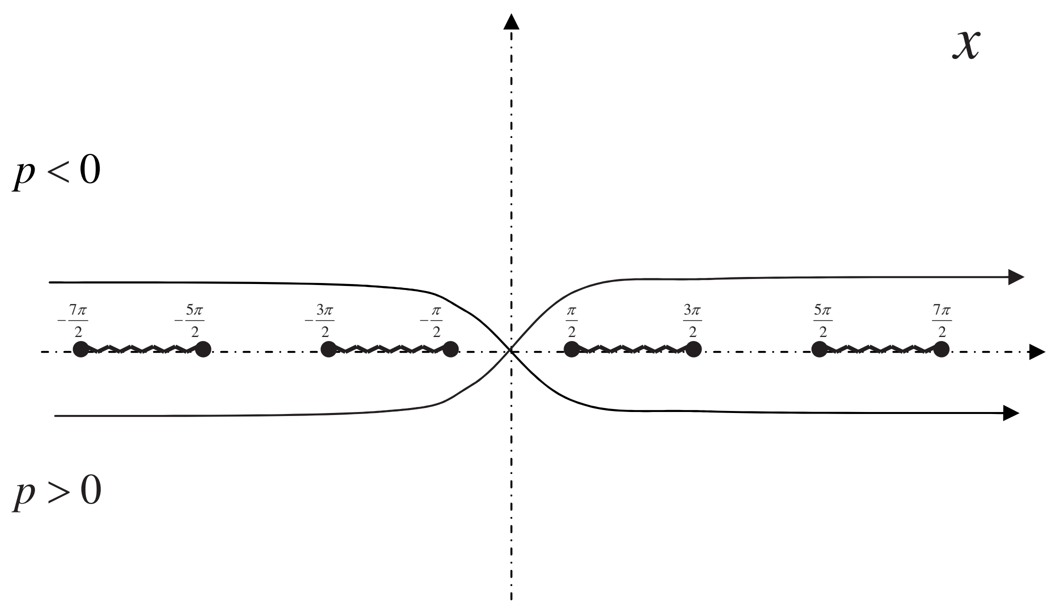
3.6 Correct Wick rotation 2

The reason why we failed in the **naive Wick rotation** is that x direction is effectively **compact** while t is **noncompact**.

→ Need to align many space-like branes (cf: rolling tachyon)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\,(2\cos x)^{-ip-1}e^{iqx} = \frac{\pi\Gamma(-ip)}{\Gamma(\frac{1}{2}-\frac{q}{2}-i\frac{p}{2})\Gamma(\frac{1}{2}+\frac{q}{2}-i\frac{p}{2})}.$$

$$\int_C dx\,[2\cos x]^{-ip-1}e^{iqx}$$
$$= \frac{\mathrm{sgn}(p)\sinh(\pi p)}{2\cosh[\frac{\pi}{2}(p+iq)]\cosh[\frac{\pi}{2}(p-iq)]} \times \frac{\pi\Gamma(-ip)}{\Gamma(\frac{1}{2}-\frac{q}{2}-i\frac{p}{2})\Gamma(\frac{1}{2}+\frac{q}{2}-i\frac{p}{2})}.$$



3.7 Density of states

Density of states is self-dual and positive definite satisfying Cardy condition.

$$\rho^{(NS)}(p',\omega') = \int_{-\infty}^{\infty} d\omega \int_{0}^{\infty} dp \, \cos(2\pi p p') \cos(2\pi \omega \omega') \frac{2\sinh\left(\frac{2\pi p}{\mathcal{Q}}\right)}{\left[\cosh\left(\frac{2\pi\omega}{\mathcal{Q}}\right) + \cosh\left(\frac{2\pi p}{\mathcal{Q}}\right)\right] \sinh(\pi \mathcal{Q} p)}$$

$$= \frac{2\sinh\left(\frac{2\pi\omega'}{\mathcal{Q}}\right)}{\left[\cosh\left(\frac{2\pi\omega'}{\mathcal{Q}}\right) + \cosh\left(\frac{2\pi p'}{\mathcal{Q}}\right)\right] \sinh(\pi \mathcal{Q} \omega')},$$

Remark 5

- R-sector has a similar structure (spectral flow).
- R-sector has a light cone divergence $p = \pm\omega$.

$$|\Psi^{(R)}(p,\omega)|^2 = \frac{\omega^2 + p^2}{(\omega^2 - p^2)} \times \frac{2\sinh\left(\frac{2\pi p}{\mathcal{Q}}\right)}{\left[\cosh\left(\frac{2\pi\omega}{\mathcal{Q}}\right) - \cosh\left(\frac{2\pi p}{\mathcal{Q}}\right)\right] \sinh(\pi \mathcal{Q} p)}.$$

- Maybe explained from the linear dilaton.

3.8 Evaluation of decay rate

We can study decay rate of rolling brane.
Emission rate shows power-like behavior
as unstable D-brane in flat space.

The decaying rate is

$$\bar{N}^{(NS)}(M) = \int_0^\infty \frac{dp}{\omega_p} \frac{2\sinh\left(\frac{2\pi p}{\mathcal{Q}}\right)}{\left[\cosh\left(\frac{2\pi\omega_p}{\mathcal{Q}}\right) + \cosh\left(\frac{2\pi p}{\mathcal{Q}}\right)\right]\sinh(\pi\mathcal{Q}p)}$$

$$\omega_p = \sqrt{p^2 + M^2} \,.$$

If we consider Dp-brane, we have ($d = 5 - p$)

$$\bar{N}(M) \sim M^{\frac{-1+d}{2}} e^{-2\pi M\sqrt{1-\frac{\mathcal{Q}^2}{4}}}$$

Combined with the density of states $n(M) = M^{-3} e^{2\pi M\sqrt{1-\frac{\mathcal{Q}^2}{4}}}$,

$$\frac{\bar{N}}{V} \sim \int^\infty dM M^{-\frac{p}{2}-1} \,, \quad \frac{\bar{E}}{V} \sim \int^\infty dM M^{-\frac{p}{2}} \,.$$

The decaying rate is exactly same as that of the rolling tachyon in flat Minkowski space (LLM).

Remark 6

- **This is expected from Kutasov's argument that two theories can be mapped to each other.**
- **Even if we have a linear dilaton, emission rate compensates unlike UV finite decay.**
- **If we analytically continue it to $\mathcal{Q} > \sqrt{2}$, emitted energy is finite!**
- **Especially, in the case of 2D black hole, radiation from the rolling D-brane is finite (now under investigation).**

Summary of this section

- **Non-BPS hairpin brane in NS5 background is Class 2 brane.**

  (hep-th/0406173)

- **Rolling brane is a Wick rotation of hairpin brane.**

- **Naive Wick rotation does not work.**

- **We performed a sophisticated Wick rotation.**

  (hep-th/0406173)

- **Emission rate is also studied.**

Some Future Works

- Application to 2D (toplogical/physical) black hole.
- Sen's conserving charge and W_∞ algebra of the rolling D-brane.
- ($\mathcal{N} = 2$) Liouville theory in higher genus?
- Topological twist of the $\mathcal{N} = 2$ LFT and coupling to D-branes.
- Connection to various matrix models and non-perturbative formulation.