Noncommutative p-Tachyon

Debashis Ghoshal

University of Tokyo, Japan
&
Harish-Chandra Res. Inst, India

Based on
DG, hep-th/0406259 (JHEP, to appear)
DG & T. Kawano, hep-th/0409311

Oct 6, 2004

Plan of the talk:

1. Introduction & motivation
2. Brief review of p-adic string theory
3. Noncommutative p-soliton
   + $p \to 1$ Limit & BSFT
   + Comments on multisolitons
4. Constant B-field in the 'worldheet'
   + Green's function & formal consequences
   + Problem with projective invariance
   + Tachyon scattering amplitudes
5. Summary & some remarks

Noncommutative deformations of quantum field theories have been studied during the past few years. The deformed theories exhibit rather intersting perturbative and nonperturbative behaviour.

Local field theory $\stackrel{\text{nc def}}{\longrightarrow}$ Non-local field theory

Some (effective) field theories are nonlocal to start with. We shall study one such.

It is the effective field theory of a scalar field: the tachyonic scalar of the so called p-adic string theory.
(Freund & Olson; Freund & Witten; Brekke, Freund, Olson & Witten; Frampton & Okada; Frampton, Okada & Ubriaco; $\cdots$)
(Also Volovic; Grossmann; $\cdots$)

The p-adic string theory is an interesting theory. (A sector) of this theory is amenable to exact calculations.

It shares many qualitative properties of usual string theory. For example, one can check the Sen conjectures for the open string tachyon in the *exact* effective field theory.

Therefore, p-adic string theory could be a useful guide to difficult questions in (usual) string theory.

However, this requires a better understanding of the p-adic string itself. We only know some properties of its D-branes in flat spacetime.

What about putting p-adic strings in non-trivial backgrounds?

For example,

- in curved spacetime: from closed string condensate;

- in electromagnetic field: from open string background.

A particularly simple background: a constant background of the rank two antisymmetric tensor field B.

It provides a noncommutative deformation of the effective field theory in spacetime.
(Schomerus; Witten; $\cdots$)

Assumption (for the first part):
the same happens for p-adic string theory.

Abbreviations

p-tachyon : Tachyon of p-adic string theory

(*Cf: Ptolemy, Pterodactyl and Ptennisnet!*)

Similarly:

p-string : p-adic string
p-soliton : Soliton of p-string theory

Review of p-string theory

(à la Brekke, Freund, Olson & Witten (1987))

Recall tree level scattering amplitude of N on-shell tachyons of momenta k_i $(i = 1, \cdots, N)$, $k_i^2 = 2, \sum k_i = 0$ is:

$$
\begin{aligned}
\mathcal{A}_N(\{k_i\}) &= \int d\xi_1 \cdots d\xi_{N-3} \\
&\times \prod_{i=1}^{N-3} |\xi_i|^{k_N \cdot k_i} |1 - \xi_i|^{k_{N-1} \cdot k_i} \\
&\times \prod_{1 \le i < j \le N-3} \left|\xi_i - \xi_j\right|^{k_i \cdot k_j} .
\end{aligned}
$$

The integrals are over the real line $\mathbf{R}$ with its translationally invariant measure $d\xi$. Integrand involves absolute values of real numbers.

The 4-point amplitude can be computed exactly in terms of Euler beta function. But $\mathcal{A}_N$ for $N \geq 5$ cannot be evaluated analytically.

The key observation of *BFOW*: all the ingredients of the amplitude $\mathcal{A}_N$ have p-adic analogues.

Proposal: Define p-string theory as follows:

- replace the absolute value norm $|\cdot|$ of $\mathbf{R}$ in the integrand by the p-adic norm $|\cdot|_p$ of $\mathbf{Q}_p$:

$$|\cdot| \to |\cdot|_p,$$

- replace integrals over $\mathbf{R}$ by integrals over $\mathbf{Q}_p$ (using the latter's translationally invariant measure).

The consequence of this is amazing:
All N-point amplitudes for p-tachyons can now be computed exactly.
I.e., we know the effective field theory of the tachyonic scalar field of p-string theory *exactly*.

\begin{digression}

Consider the *rational numbers* $\mathbf{Q}$. We are familiar with the *absolute value norm*. Now we will define different norms on this field.

1. Fix a prime number $p \in \{2, 3, 5, 7, 11, 13, \cdots\}$,

2. For any integer z, find the *highest* power of p that divides it. Call it $\text{ord}_p z$.

   $\text{ord}_p \sim$ logarithm: $\text{ord}_p(z_1 z_2) = \text{ord}_p z_1 + \text{ord}_p z_1$.

3. For a rational number $q = \frac{z_1}{z_2}$:
   $\text{ord}_p q = \text{ord}_p z_1 - \text{ord}_p z_2$.

4. The *p-adic norm* is:

$$|q|_p = \begin{cases} p^{-\text{ord}_p q}, & q \neq 0 \\ 0, & q = 0 \end{cases}$$

Examples: $|5|_5 = \frac{1}{5}$, $|35|_5 = \frac{1}{5}$, $|250|_5 = \frac{1}{5^3} = \frac{1}{125}$, $|6|_5 = 1$, $\left|\frac{2}{3}\right|_5 = 1$, $\left|\frac{2}{5}\right|_5 = 5$, $|96|_2 = \frac{1}{32}$.

$|q|_p$ satisfies all the properties of a norm. In fact there is a stronger triangle inequality:

$$|q_1 + q_2|_p \leq \max \left\{|q_1|_p, |q_2|_p\right\}$$

This norm is called *non-Archimedian* and leads to many strange properties:

- All triangles are *isosceles*.

- Any point in the *interior* of a disc is a *centre*.

- If $|q_1|_p < |q_2|_p$, then $|n\, q_1|_p < |q_2|_p$ for any $n \in \mathbf{Z}$.

Now recall that the field of real number **R** is obtained from **Q** adding to **Q** the *limit points* of Cauchy sequences. The limit points are determined by the absolute value norm.

Cauchy completion of **Q** by the p-adic norm leads to a new field of numbers $\mathbf{Q}_p$

Any number in $\mathbf{Q}_p$ can be represented as a power series in p

$$\xi = p^N \left(\xi_0 + \xi_1 p + \xi_2 p^2 + \cdots\right)$$

where, $\xi_n = \{0, 1, 2, \cdots, p-1\}$, $\xi_0 \neq 0$.

E.g. in $\mathbf{Q}_7$, $\frac{1}{2} = 4 + 3.7 + 3.7^2 + 3.7^3 + \cdots$.

This is like the Laurent series expansion.

Elements of $\mathbf{Q}_p$ with norm at most 1, form a maximal compact subring:

$$\mathbf{Z}_p = \{\xi \in \mathbf{Q}_p : \ |\xi|_p \leq 1\},$$

the ring of p-adic integers. Ordinary integers are a dense subset in it.

Integration in $\mathbf{Q}_p$: There is *translationally invariant* real valued Haar measure dx.
Examples:

- Normalization: $\int_{\mathbf{Z}_p} d\xi = 1$.

- Gelfand-Graev-Tate gamma-function:

$$\Gamma(s) = \int_{\mathbf{Q}_p} d\xi \, e^{2\pi i \{\xi\}} \, |\xi|_p^{s-1} = \frac{1 - p^{s-1}}{1 - p^{-s}}$$

[Cf: $\Gamma_{\mathrm{R}}(s) = \int_{\mathrm{R}} dx \, e^{2\pi i \{x\}} \, |x|^{s-1} = \frac{2\cos(\pi s/2)}{(2\pi)^s} \Gamma_{\text{Euler}}(s)$.]
and beta-function:

$$\int_{\mathbf{Q}_p} d\xi \, |\xi|_p^{x-1} |1 - \xi|_p^{y-1} = \Gamma(x)\Gamma(y)\Gamma(1-x-y)$$

\end{digression}

Dynamics of the p-tachyon field is summarized by its *spacetime* effective lagrangian:

$$\mathcal{L}^{(p)} = \frac{p^2}{g^2(p-1)} \left[-\frac{1}{2}\varphi p^{-\frac{1}{2}\Box}\varphi + \frac{1}{p+1}\varphi^{p+1} \right].$$

The above is written in terms of

$$\varphi(x) = 1 + \frac{g}{p}T(x),$$

T is the original p-tachyon.

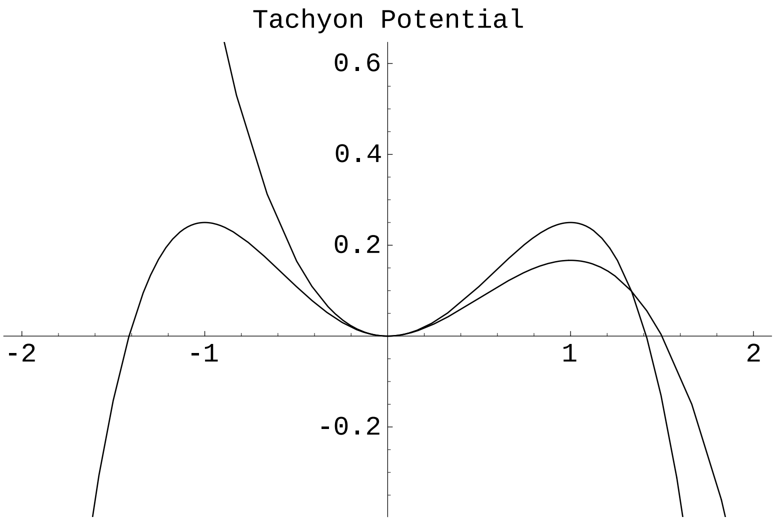
Perturbative vacuum $T = 0$ correspond to $\varphi = 1$.

Parenthetic remark: Although derived for a prime p, the spacetime action makes sense for all integer values of p.

Kinetic term in is non-standard and involves an infinite number of derivatives.

Potential has a local minimum and two (respectively one) local maxima for odd (respectively even) integer p:

p-tachyon potential



Potential always has runaway pathological singularities.

Equation of motion:

$$p^{-\frac{1}{2}\Box}\varphi = \varphi^p.$$

The solutions to these are:

- The *constant* configuration $\varphi(x) = 1$.
  This is the vacuum around which we have quantized — a space-filling D-25-brane.
  The *tension* of the brane is

$$\mathcal{T}_{25} = -\mathcal{L}(\phi = 1) = \frac{1}{2g^2}\frac{p^2}{p+1}.$$

- The *constant* configuration $\varphi(x) = 0$.
  There are *no* perturbative excitation around this configuration. Energy vanishes in this configuration.
  Closed string vacuum?

- Space(time) dependent solutions:

The eqn of motion is *separable* in orthogonal coordinates.

$$p^{-\frac{1}{2}\partial_y^2} f(y) = f^p(y)$$

This has a non-trivial solution:

$$f(y) = p^{1/2p(p-1)} \exp\left(-\frac{p-1}{2p \ln p} y^2\right).$$

Gaussian lump with correlated amplitude and spread.

The configurations

$$\varphi(x) = f(x^{m+1}) f(x^{m+2}) \cdots f(x^{25}) = F^{(m)}(x_\perp)$$

is a solution to the eqn of motion for $m = 0, 1, \cdots, 24$.

$$(\underbrace{x^0, x^1, \cdots, x^m}_{x_\parallel}, \underbrace{x^{m+1}, \cdots, x^{25}}_{x_\perp})$$

$\varphi(x) = F^{(m)}(x_\perp)$ describes a *lump* with energy *localised* around the hyperplane $x_\perp = 0$.

This is a solitonic m-brane.

Its tension (=energy/volume) is

$$\mathcal{T}_m = \frac{p^2}{2g^2(p+1)} \left[\frac{2\pi p^{2p/(p-1)} \ln p}{p^2 - 1} \right]^{(25-m)/2}$$
$$\equiv \frac{1}{2g_m^2} \frac{p^2}{p+1}.$$

Ratio of tension $\frac{\mathcal{T}_m}{\mathcal{T}_{m-1}}$ is independent of m: as in ordinary string theory.

Can study the worldvolume theory on the soliton. A consistent truncation keeping only the tachyon is possible. There are also massless fields corresponding to translation zero modes.

p-Tachyon field behaves according to the conjectures of Sen. (DG & Sen, 2000)

Noncommutative field theories

Spacetime coordinates do not commute:

$$[x^i, x^j] = i\theta\epsilon^{ij}.$$

For simplicity, $i, j = 1, 2$ and $\theta =$real constant.

In string theory, the deformation arises from a constant background of the two-form antisymmetric tensor field B along x^1-x^2.

An equivalent description: use commuting coordinates, but while multiplying fields, which are dependent on x^1 and x^2, use the Moyal star product

$$f \star g = f(x^1, x^2) \exp\left(\frac{i}{2}\theta\epsilon^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right) g(x^1, x^2),$$

instead of ordinary pointwise multiplication.

Alternatively, f, g are operator-valued functions through the Moyal-Weyl correspondence.

(Commutative) field theories may be deformed by using Moyal star product in the action, equations of motion, etc.

Example: theory of a single scalar field ϕ:

$$\mathcal{L}_{NC} = \frac{1}{2}(\partial_\mu \phi) \star (\partial^\mu \phi) - \mathcal{V}(\star\phi),$$

where, $\mathcal{V}(\star\phi)$ may be, *e.g.*,

$$\mathcal{V}(\star\phi) \sim \frac{m^2}{2}\phi\star\phi + \frac{g}{3!}\phi\star\phi\star\phi + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi + \cdots.$$

Noncommutative field theories exhibit interesting perturbative and nonperturbative properties.

Noncommutative scalar field theory has *soliton* solutions in the limit of infinite noncommutativity: $\theta \to \infty$.
(Gopakumar, Minwalla & Strominger; $\cdots$)

The simplest of the solitons is a gaussian lump whose width and amplitude are fixed:

$$\phi(x^1, x^2) = 2 \exp\left(-\frac{(x^1)^2 + (x^2)^2}{\theta} \right). \qquad (1)$$

More general solutions correspond to the solutions of projector equation:

$$\pi \star \pi \sim \pi.$$

However, these solitons become unstable at finite θ:
Higher projectors are unstable and decay to the gaussian lump,
Gaussian lump becomes unstable below a critical value of θ.

These (and other) noncommutative solitons have been used in string (field) theory to understand the dynamics of the tachyon. (Dasgupta, Mukhi & Rajesh; Harvey, Kraus, Larsen & Martinec; $\cdots$)

Noncommutative deformation of the p-tachyon

Use $\star$-product in the p-tachyon action:

$$\mathcal{L}_{NC}^{(p)} = -\varphi \star p^{-\frac{1}{2}\Box}\varphi + \frac{1}{p+1}(\star\varphi)^{p+1},$$

We define the noncommutative p-tachyon by the above action.

Equation of motion:

$$p^{-\frac{1}{2}\Box}\varphi = (\star\varphi)^{p}.$$

We will be interested only in the part dependent on x^1 and x^2.

The solutions, $\varphi = 0, 1$ describing constant configurations, are still solutions of the deformed eom.

There are nontrivial solutions: these are gaussian solitonic lumps.

Observe: $\star$-product of gaussian is again a gaussian:

$$Ae^{-a|z|^2} \star Be^{-b|z|^2} = \frac{AB}{1+ab\theta^2} \exp\left(\frac{a+b}{1+ab\theta^2}|z|^2\right).$$

Make a gaussian ansatz:

$$\varphi(x^1, x^2) = A^2 \exp\left(-a\left[(x^1)^2 + (x^2)^2\right]\right)$$

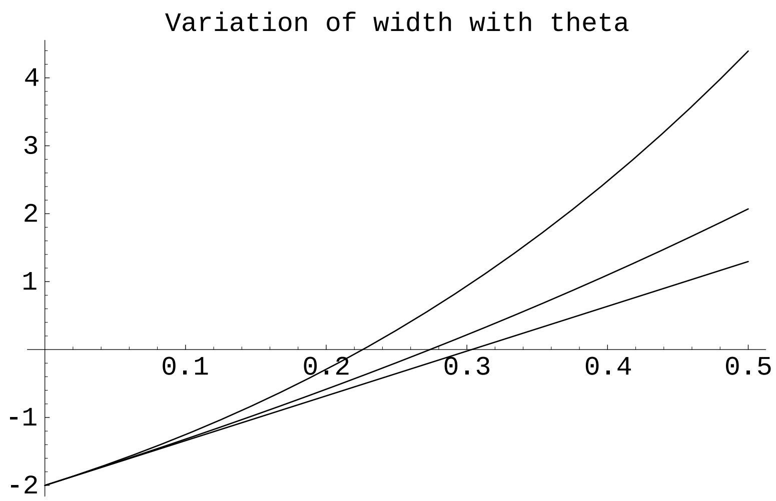
and equate width a and amplitude A on both sides of eom.

Spread a is determined by a polynomial equation of degree p:

$$\sum_{i=0}^{\lfloor p/2 \rfloor} \binom{p}{2i} (a\theta)^{2i} - (1 - 2a \ln p) \sum_{i=0}^{\lfloor (p-1)/2 \rfloor} \binom{p}{2i+1} (a\theta)^{2i} = 0,$$

For an odd p, there is always one real root. For $p = 2$, the roots are real, with the positive root being the relevant one.

The polynomial for $p = 3$

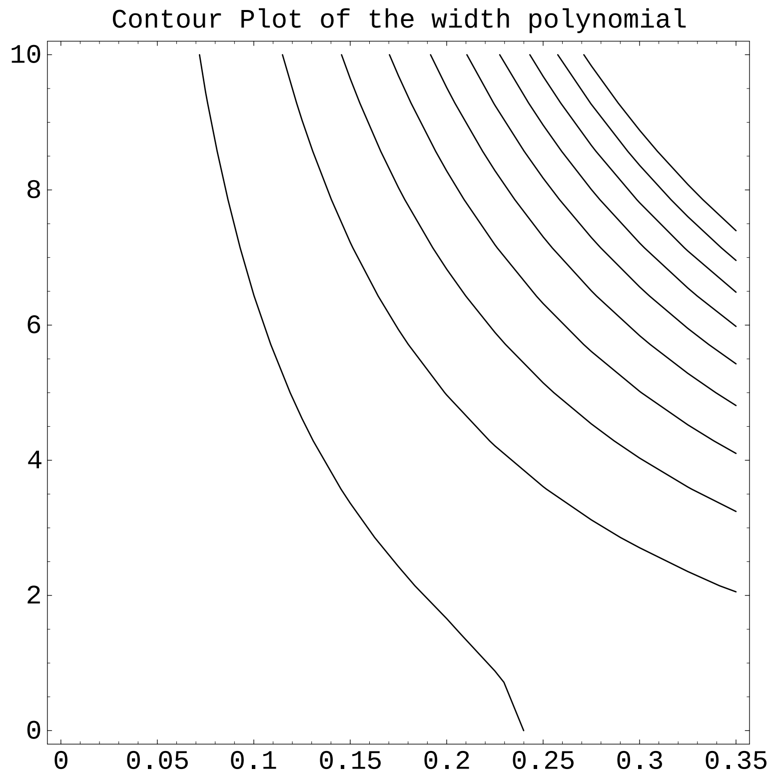


Limiting cases:

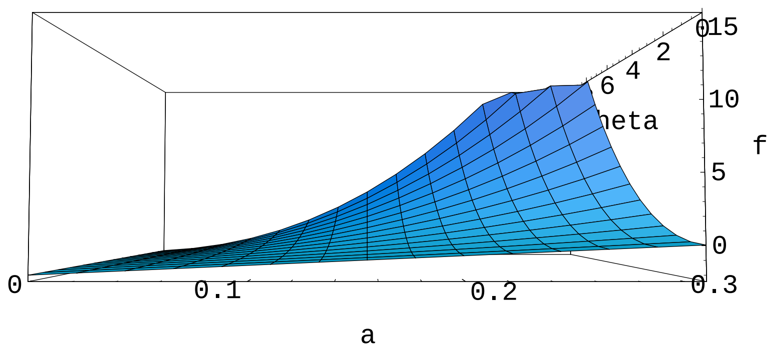
- Commutative limit: $\theta = 0$
  The width of the commutative gaussian lump of p-string theory $a = (p-1)/2p \ln p$ is recovered.

- Large noncommutative limit: $\theta \to \infty$
  Keeping only the term of highest degree in a naively is not the right thing. Indeed using $a \sim 1/\theta$ for large θ, this is subdominant. Correct treatment gives $a = 1/\theta$.

Width as a function of θ



Polynomial as a function of a and θ



Amplitude A is determined in terms of the width:

$$A = \left[\sum_{i=0}^{\lfloor (p-1)/2 \rfloor} \binom{p}{2i+1} (a\theta)^{2i} \right]^{1/2(p-1)} .$$

This, again, interpolates between

- Commutative limit ($\theta = 0$):
  $A = p^{1/2(p-1)}$,

- Large noncommutativity limit ($\theta \to \infty$):
  $A = \sqrt{2}$.

Summary:

Noncommutative p-soliton interpolates smoothly between noncommutative soliton and commutative p-soliton.

$p \to 1$ Limit & BSFT

In this limit the lagrangian of p-string theory reduces to:

$$\mathcal{L}^{(p\to 1)} = -\frac{1}{2}\varphi\,\Box\,\varphi - \frac{1}{2}\varphi^2\left(\ln\varphi^2 - 1\right).$$

This is exactly the boundary string field theory (BSFT) action for the tachyon of ordinary bosonic string, truncated to two derivatives! (Gerasimov & Shatashvili; Kutasov, Marino & Moore)

Introduce noncommutativity [Cornalba, Okuyama]:

$$\mathcal{L}^{(p\to 1)}_{NC} = -\frac{1}{2}\varphi\star\Box\,\varphi - \frac{1}{2}\varphi\star\varphi\left(\ln_\star(\varphi\star\varphi) - 1\right)$$

yields the equation of motion:

$$\Box\,\varphi + 2\varphi\star\ln_\star\varphi = 0.$$

Solutions:

- Constant configurations $\varphi = 0, 1$,
- Deformation of gaussian lump of BSFT?

Gaussian ansatz: LHS is exactly as in the commutative case.

$$\square\varphi = 4a(a|z|^2 - 1)\, A^2 \exp(-a|z|^2).$$

RHS is less obvious, since we need the $\star$-deformed logarithm of an ordinary exponential.

From the n-fold $\star$-product of the gaussian, we can extrapolate to fractional powers.

For $g(z) = Ae^{-a|z|^2}$:

$$\lim_{\epsilon \to 0} \left[\frac{1}{\epsilon} (\star g(z))^{1+\epsilon} - \frac{1}{\epsilon} \right]$$

$$= \Big[\left(2 \ln A - \frac{(a\theta)^2}{1.2} - \frac{(a\theta)^4}{3.4} - \cdots \right) -2a|z|^2 \left(\frac{1}{2} - \frac{(a\theta)^2}{1.3} - \frac{(a\theta)^4}{3.5} - \cdots \right) \Big] A^2 e^{-a|z|^2}$$

Hence, by comparison

$$2a = \frac{1-(a\theta)^2}{2a\theta} \ln \left(\frac{1+a\theta}{1-a\theta} \right)$$

$$2 \ln A = \frac{1+a\theta}{2a\theta} \ln(1+a\theta) - \frac{1-a\theta}{2a\theta} \ln(1-a\theta)$$

Width a and amplitude A are determined by *transcendental* equation.

They interpolate smoothly between the BSFT solution $a = \frac{1}{2}$, $A = \sqrt{e}$ to the GMS solution $a = \frac{1}{\theta}$, $A = \sqrt{2}$.

Multisolitons

Consider

$$f_{w_1 w_2} = \exp\left(-\frac{1}{\theta}(\bar{z} - \bar{w}_1)(z - w_2)\right).$$

The configuration

$$\sum_{i,j=1}^{n} A_{ij} f_{w_i w_j} = \sum_{i,j} A_{ij} e^{-(\bar{z}-\bar{w}_i)(z-w_j)/\theta}$$

solves the eom $\pi \star \pi \sim \pi$ at $\theta \to \infty$ for specific choices of A_{ij}.
(Headrick, Gopakumar & Spradlin)

By Moyal-Weyl correspondence

$$f_{w_i w_j} \sim |w_i\rangle\langle w_j|,$$

where,

$$|w\rangle \sim e^{w a^\dagger}|0\rangle$$

is a coherent state. The amplitudes of a multisoliton are

$$||A_{ij}|| \sim ||\langle w_i | w_j \rangle||^{-1}.$$

More generally, define:

$$f_{w_i w_j}(a_{ij}, A_{ij}) = A_{ij} \exp\Big(-a_{ij}(\bar{z}-\bar{w}_i)(z-w_j)\Big).$$

The LHS of the eom of the p-tachyon:

$$\begin{aligned} & e^{-\frac{1}{2}\ln p \Box} f_{w_i w_j}(a_{ij}, A_{ij}) \\ = \; & f_{w_i w_j}\left(\frac{a_{ij}}{1+2a_{ij}\ln p}, \frac{A_{ij}}{1+2a_{ij}\ln p}\right). \end{aligned}$$

However, at arbitrary θ, the RHS involves $\star$-product of these $f_{w_i w_j}(a_{ij}, A_{ij})$. This is given by:

$$f_{w_1 w_2}(a_{12}, A_{12}) \star f_{w_3 w_4}(a_{34}, A_{34}) = f_{\tilde{u}\tilde{v}}(\tilde{a}, \tilde{A}),$$

where, the modified width $\tilde{a}$, amplitude $\tilde{A}$ and the centres $\tilde{u}$ and $\tilde{v}$ are given by:

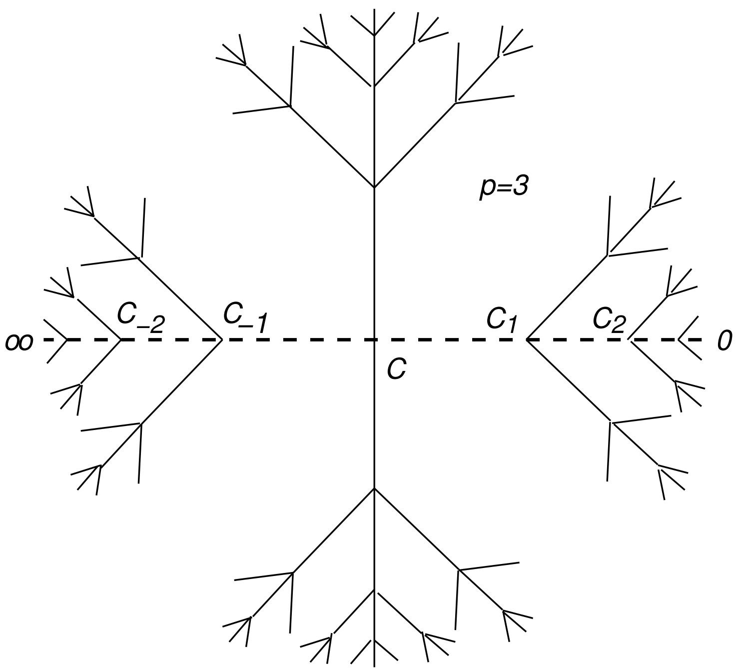
$$
\begin{aligned}
\tilde{a} &= \frac{a_{12}+a_{34}}{1+\theta^2 a_{12}a_{34}} \\
\tilde{A} &= \frac{A_{12}+A_{34}}{1+\theta^2 a_{12}a_{34}} \\
&\quad \times \ \exp\left\{-\frac{a_{12}a_{34}}{a_{12}+a_{34}}(\bar{w}_1-\bar{w}_3)(w_2-w_4)\right\} \\
\bar{\tilde{u}} &= \frac{1}{a_{12}+a_{34}}\Big[a_{12}(1+a_{34}\theta)\bar{w}_1 \\
&\qquad +a_{34}(1-a_{12}\theta)\bar{w}_3\Big] \\
\tilde{v} &= \frac{1}{a_{12}+a_{34}}\Big[a_{12}(1-a_{34}\theta)w_2 \\
&\qquad +a_{34}(1+a_{12}\theta)w_4\Big].
\end{aligned}
$$

So, not only do the width and the amplitude change, the centres are also shifted.

Noncommutativity from B-field?

Question: Is there a microscopic (worldsheet) understanding of the noncommutativity?

Recall that the worldsheet of the p-adic string is an infinite *tree*, a *graph* without any loop. In other words, it is a *Bethe lattice* with $(p+1)$ nearest neighbours. [Zabrodin *et al.*]



Wordsheet of the 3-adic string:

The tree $\mathcal{T}_p$ for $p = 3$.

The boundary of the tree $\mathcal{T}_p$ is $\mathbf{Q}_p$ (through the power series expansion

$$\xi = p^N(\xi_0 + \xi_1 p + \xi_2 p^2 + \cdots),$$

$\xi_n \in \{0, 1, \cdots, p-1\}, \xi_0 \neq 0.$)

The usual string worldsheet is the strip which is conformally equivalent to the UHP = SL(2,**R**)/SO(2,**R**).
SL(2,**R**) acts on the boundary of the worldsheet as a symmetry, (and SO(2) is its maximal compact subgroup).

The symmetry group of the boundary of the p-adic worldsheet is PGL(2,$\mathbf{Q}_p$).
PGL(2,$\mathbf{Q}_p$)/PGL(2,$\mathbf{Z}_p$) = $\mathcal{T}_p$

The 'worldsheet' of the p-adic string is *discrete*, but its boundary is a *continuum*.

p-adic string provides an *exotic discretisation* of the open string worldsheet.

Polyakov action is the discrete lattice action:

$$S_p(X) = \frac{1}{2}\beta_p \sum_{\mathbf{e}} (\delta_{\mathbf{e}} X^\mu)(\delta_{\mathbf{e}} X^\nu)\eta_{\mu\nu}$$

The eqn of motion is the discrete Laplace eqn. Both *Neumann* and *Dirichlet* boundary conditions lead to well defined boundary value problem. Hence, one can have D-branes.

Integrate over the degrees of freedom in the bulk of $\mathcal{T}_p$ to get a non-local action on the boundary [Spokoiny, Zhang, Parisi]:

$$\mathcal{S}_p = \frac{p(p-1)\beta_p}{4(p+1)} \int_{\mathbf{Q}_p} d\xi\, d\xi' \, \frac{(X^\mu(\xi) - X^\mu(\xi'))^2}{|\xi - \xi'|_p^2}.$$

This form will be more useful to us, since a constant B-field couple to the boundary. For the usual string, after an integration by parts:

$$\int_{\partial\Sigma} d\xi\, B_{\mu\nu} X^\mu(\xi) \partial_t X^\nu(\xi)$$

Formally same as the insertion of a background gauge field $A_\mu [X(\xi)] \sim B_{\mu\nu} X^\nu(\xi)$.

Now the problem is with a tangential derivative. Get around this through the Cauchy-Riemann relation for harmonic/ analytic functions. Define:

$$\partial_t^{(p)} X^\mu(\xi) \equiv \partial_\xi X^\mu = \int_{\mathbf{Q}_p} d\xi' \, \frac{\mathrm{sgn}_\tau(\xi - \xi')}{|\xi - \xi'|_p^2} \, X^\mu(\xi')$$

\begin{digression}

The field $\mathbf{Q}_p$ (like **R**) is *not* algebraically closed. Not all $\xi \in \mathbf{Q}_p$ has a square-root in it.

Recall, quadratic extension of **R** is $\mathbf{R}(\sqrt{-1}) = \mathbf{C}$.

Three inequivalent quadratic extension of $\mathbf{Q}_p$: $\mathbf{Q}_p(\sqrt{\tau})$ for $\tau = \varepsilon, p$ and $\tau = \varepsilon p$ $(p \neq 2)$, where ε is a $(p-1)$-th root of unity.

Define a (real valued) function on $\mathbf{Q}_p$ (*not* on its extension):

$$\mathrm{sgn}_\tau(\xi) = \begin{cases} +1, & \text{if } \xi = \zeta_1^2 - \tau\zeta_2^2 \\ & \text{for some } \zeta_1, \zeta_2 \in \mathbf{Q}_p, \\ -1 & \text{otherwise} \end{cases}$$

Moreover, if we demand antisymmetry

$$\mathrm{sgn}_\tau(-\xi) = -\,\mathrm{sgn}_\tau(\xi)$$

we must restrict to

$$\begin{aligned} \tau &= p,\ \varepsilon p \\ p &= 3 \pmod 4 \end{aligned}$$

Generalised gamma-function:

$$\begin{aligned} \Gamma_\tau(s) &= \int_{\mathbf{Q}_p} d\xi\, e^{2\pi i\xi}\, |\xi|_p^{s-1} \mathrm{sgn}_\tau(\xi) \\ &= \pm\sqrt{\mathrm{sgn}_\tau(-1)}\, p^{s-\frac{1}{2}} \end{aligned}$$

\end{digression}

Proposed action with B-field

$$\int_{\mathbf{Q}_p} \frac{\eta_{\mu\nu}\left(X^\mu(\xi) - X^\mu(\xi')\right)\left(X^\nu(\xi) - X^\nu(\xi')\right)}{|\xi - \xi'|_p^2} d\xi d\xi'$$
$$+ \frac{i(p+1)}{p^2 \Gamma_\tau(-1)} \int_{\mathbf{Q}_p} B_{\mu\nu}\, X^\mu(\xi) \frac{\mathrm{sgn}_\tau(\xi - \xi')}{|\xi - \xi'|_p^2} X^\nu(\xi') d\xi d\xi'$$

Easy to solve for the Green's function in the presence of the B-field:

$$\mathcal{G}^{\mu\nu}\Big(\xi - \xi'\Big) = -G^{\mu\nu} \ln|\xi - \xi'|_p + \frac{i}{2}\theta^{\mu\nu} \mathrm{sgn}_\tau\left(\xi - \xi'\right),$$
$$\left(\frac{1}{\eta - iB}\right) = G^{-1} + \frac{i}{2}\frac{p-1}{\alpha' p \ln p\, \Gamma_\tau(0)}\theta$$

Notice the similarity with the usual bosonic string [Fradkin-Tseytlin, Abouelsaood *et al*].

$$\left\langle \prod_{I=1}^{N} e^{ik^I \cdot X}(\xi_I) \right\rangle_B = \prod_{I<J} \frac{\exp\left(-\frac{i}{2} k^I \theta k^J \mathrm{sgn}_\tau(\xi_I - \xi_J)\right)}{|\xi_I - \xi_J|_p^{-k^I G k^J}}$$

[Grange, hep-th/0409305]

Formal consequence of the B-field

This may seem to be the end of the road, since:

$$
\begin{aligned}
& [X^\mu(0), X^\nu(0)] \\
= \;& T\left(\langle X^\mu(0) X^\nu(0-)\rangle - \langle X^\mu(0) X^\nu(0+)\rangle\right) \\
= \;& \lim_{\substack{\mathrm{sgn}_\tau(\xi)=-1 \\ |\xi|_p \to 0}} \langle X^\mu(0) X^\nu(\xi)\rangle \\
& \qquad - \lim_{\substack{\mathrm{sgn}_\tau(\xi)=+1 \\ |\xi|_p \to 0}} \langle X^\mu(0) X^\nu(\xi)\rangle \\
= \;& i\,\theta^{\mu\nu}
\end{aligned}
$$

but there are caveats.

There is no natural notion of an order in $\mathbf{Q}_p$.

We *can* define an order by, say, ordering the ξ_n in $\xi = p^N \sum \xi_n p^n$. We can also distinguish between positive and negative p-adic numbers.

But, this notion is not GL(2,$\mathbf{Q}_p$) covariant.

Tachyon scattering amplitudes

To establish the relation between B-field and spacetime noncommutativity, one needs the tachyon scattering amplitudes.

Three-tachyon amplitude

$$\mathcal{A}^{(3)} = \cos \frac{1}{2} k^1 \theta k^2 \equiv c_{12}$$

- There is no projective invariance. But we fix three positions by hand.
- Added and averaged over another term with $1 \leftrightarrow 2$.

This agrees with the result from noncommutative deformation of the tachyon effective field theory.

Four-tachyon amplitude

From p-adic string theory with B-field:

$$\mathcal{A}^{(4)} = \frac{p-1}{p}\left(\frac{c_{12}c_{34}}{p^{\alpha(s)}-1} + \frac{c_{13}c_{24}}{p^{\alpha(t)}-1} + \frac{c_{14}c_{23}}{p^{\alpha(u)}-1}\right) - \frac{1}{p}(c_{12}c_{34} + c_{13}c_{24} + c_{14}c_{23}) + \frac{p+1}{p}c_{12}c_{13}c_{23}$$

Compare with the result from field theory:

$$\mathcal{A}^{(4)} = \frac{p-1}{p}\left(\frac{c_{12}c_{34}}{p^{\alpha(s)}-1} + \frac{c_{13}c_{24}}{p^{\alpha(t)}-1} + \frac{c_{14}c_{23}}{p^{\alpha(u)}-1}\right) + \frac{p-2}{p}(c_{12}c_{34} + c_{13}c_{24} + c_{14}c_{23})$$

Not quite the same, unfortunately!

Summary

Studied a noncommutative deformation of the exact effective action of the p-tachyon.

Obtained a family of gaussian lump solution for all values of the noncommutativity parameter θ. Smoothly interpolates between the soliton of (commutative) p-adic string theory and the noncommutative soliton of GMS.

These solitons owe their existence to the infinite derivatives in the equation of motion.

Meaningful $p \to 1$ limit gives smoothly interpolating solitons in BSFT of usual bosonic string theory.

Coupled B-field to the nonlocal action on the boundary of the 'worldsheet' and examined its effect on the spacetime theory. But ...

Some remarks

- Our lump solutions for all θ is unlike other exact solutions. *E.g.*, Polychronakos, Harvey *et al* obtained a family of solution in a field theory of scalars and gauge fields. But it has no smooth commutative limit.

- The similarity between the solitons of p-string theory and noncommmutative field theories have been noticed before. Dragovich & Volovich noticed that the GMS soliton for $\theta = 4 \ln 2$ is identical to solitonic 2-brane of the 2-adic string theory. This is merely a coincidence and perhaps of no real significance.

- Still lacking a worldsheet understanding of the noncommutative deformation in the space-time effective action.

  We coupled the B-field to the boundary. It can be coupled to the bulk as

$$\int X_*(B) = \text{pull-back of the 2-form } B$$

  Problem is that there is no 2-cycle in the tree. However, . . .

- p-adic string in other backgrounds?
  Closed p-strings?

THANK YOU!