δN formalism for curvature perturbations from inflation

Misao Sasaki

Yukawa Institute (YITP)
Kyoto University

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- metric perturbation & time slicing
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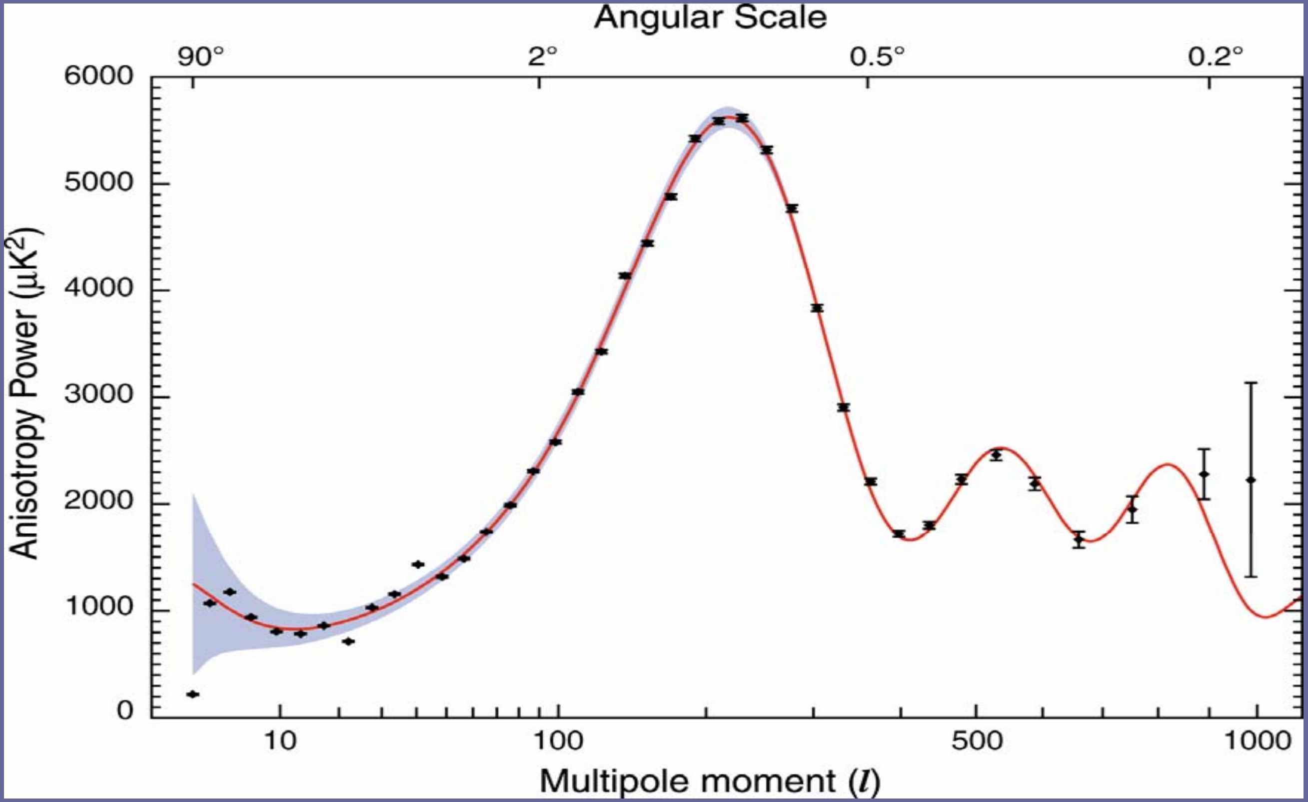
5. Summary

1. Introduction

- Standard (single-field, slowroll) inflation predicts scale-invariant Gaussian curvature perturbations.



- CMB (WMAP) is consistent with the prediction.
- Linear perturbation theory seems to be valid.

- So, why bother doing more research on inflation?

  Because observational data does not exclude other models.

  Tensor perturbations have not been detected yet.

  T/S ~ 0.2 - 0.3? or smaller?

- In fact, inflation may not be so simple.

  multi-field, non-slowroll, extra-dim's, string theory...

- PLANCK, CMBpol, ... may detect non-Gaussianity

  $\Psi = \Psi_{gauss} + f_{NL} \Psi^2_{gauss} + \cdots$; $|f_{NL}| \gtrsim 5$?

- Nonlinear backreaction on superhorizon scales?

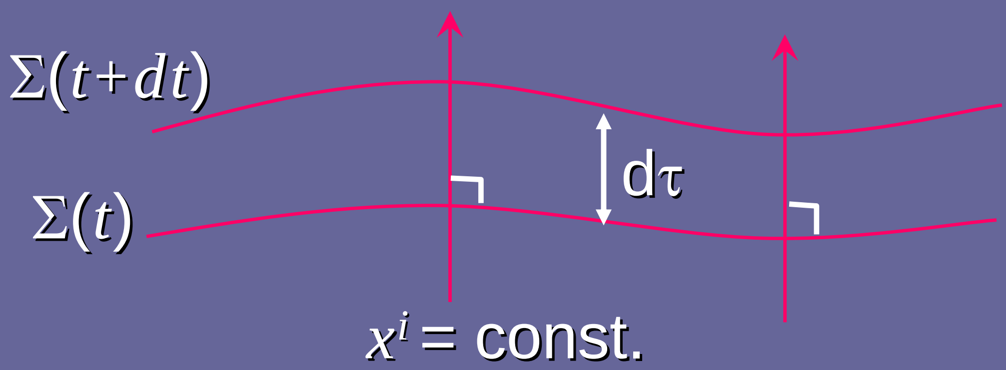
Re-consider the dynamics on super-horizon scales

2. Linear perturbation theory

Bardeen '80, Mukhanov '81, Kodama & MS '84, ….

- metric on a spatially flat background (g_{0j}=0 for simplicity)

$$ds^2 = -(1+2A)dt^2 + a^2(t)\left[(1+2\mathcal{R})\delta_{ij} + H_{ij}\right]dx^i dx^j$$



$$\left(H_{ij}\right)_{\text{scalar}} = \partial_i \partial_j E$$

$$\left(H_{ij}\right)_{\text{tensor}} = \text{transverse-traceless}$$

- propertime along x^i = const.: $d\tau = (1+A)dt$
- curvature perturbation on $\Sigma(t)$: P $\Leftrightarrow$ ${}^{(3)}R = -\dfrac{4}{a^2}\,{}^{(3)}\Delta\,\mathcal{R}$
- expansion (Hubble parameter): $\tilde{H} = H(1-A) + \partial_t\left[\mathcal{R} + \dfrac{1}{3}\,{}^{(3)}\Delta E\right]$

Choice of time-slicing

matter-based slices

- comoving slicing $T^{\mu}{}_{i} = 0 \quad \left(\phi = \phi(t) \text{ for a scalar field}\right)$
- uniform density slicing $-T^{0}{}_{0} \equiv \rho = \rho(t)$

geometrical slices

- uniform Hubble slicing

$$\tilde{H} = H(t) \Leftrightarrow -H A + \partial_t \left[\mathcal{R} + \frac{1}{3} \overset{(3)}{\Delta} E \right] = 0$$

- flat slicing $\overset{(3)}{R} = -\frac{4}{a^2} \overset{(3)}{\Delta} \mathcal{R} = 0 \Leftrightarrow \mathcal{R} = 0$

- Newton (shear-free) slicing

$$\partial_t \left[H_{ij} \right]^{\text{scalar}}_{\text{traceless}} \equiv \left[\partial_i \partial_j - \frac{1}{3} \delta_{ij} \overset{(3)}{\Delta} \right] \partial_t E = 0 \Leftrightarrow \partial_t E = 0 \Leftrightarrow E = 0$$

comoving = uniform ρ = uniform H on superhorizon scales

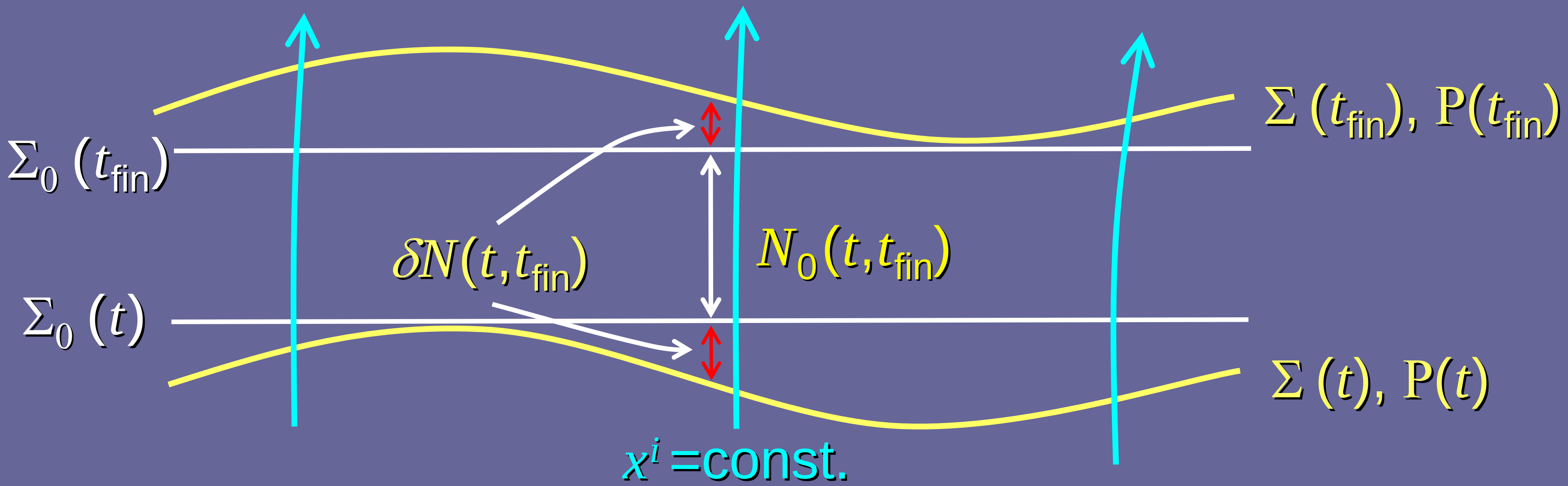
• δN formalism in linear theory

MS & Stewart '96

e-folding number perturbation between $\Sigma(t)$ and $\Sigma(t_{\rm fin})$:

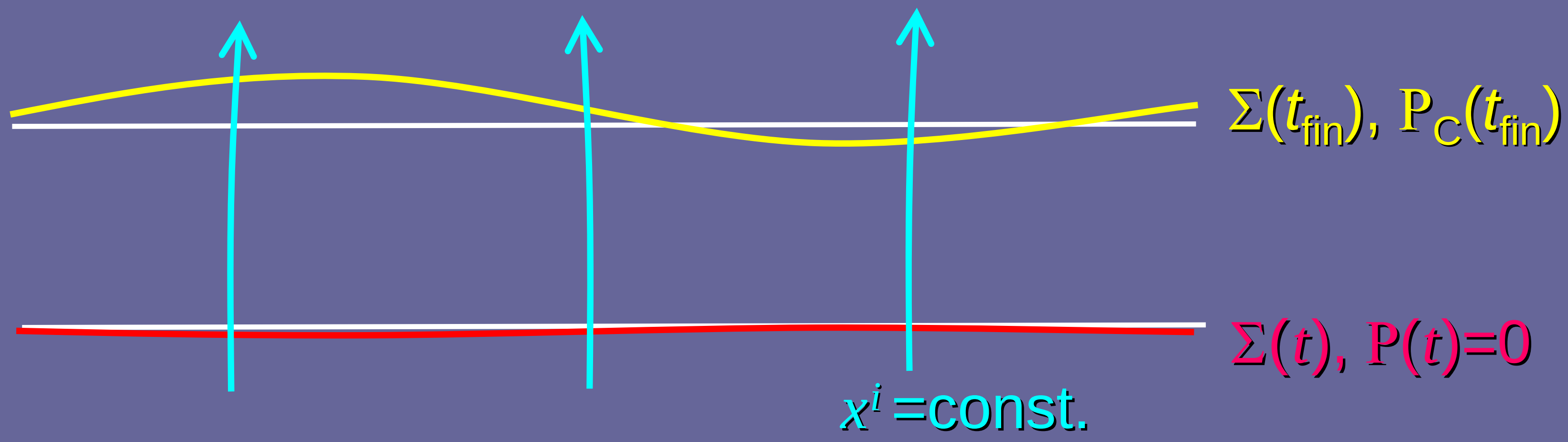
$$\delta N(t;t_{\rm fin}) \equiv \int_t^{t_{\rm fin}} H\, d\tau - \left(\int_t^{t_{\rm fin}} H\, d\tau\right)_{\rm background}$$

$$= \int_t^{t_{\rm fin}} \partial_t \left[\mathcal{R} + \frac{1}{3}\,{}^{(3)}\!\Delta E\right] dt = \mathcal{R}(t_{\rm fin}) - \mathcal{R}(t) + O(\varepsilon^2)$$



δN=0 if both $\Sigma(t)$ and $\Sigma(t_{\rm fin})$ are chosen to be 'flat' ($\mathcal{R}$=0).

Choose $\Sigma(t)$ = flat (P=0) and $\Sigma(t_{\rm fin})$ = comoving:



$$\Rightarrow \quad \delta N\left(t;t_{\rm fin}\right) = R_C\left(t_{\rm fin}\right)$$

curvature perturbation on comoving slice (suffix 'C' for comoving)

The gauge-invariant variable 'ζ' used in the literature

is related to P_C as $\zeta = -P_C$ or $\zeta = P_C$

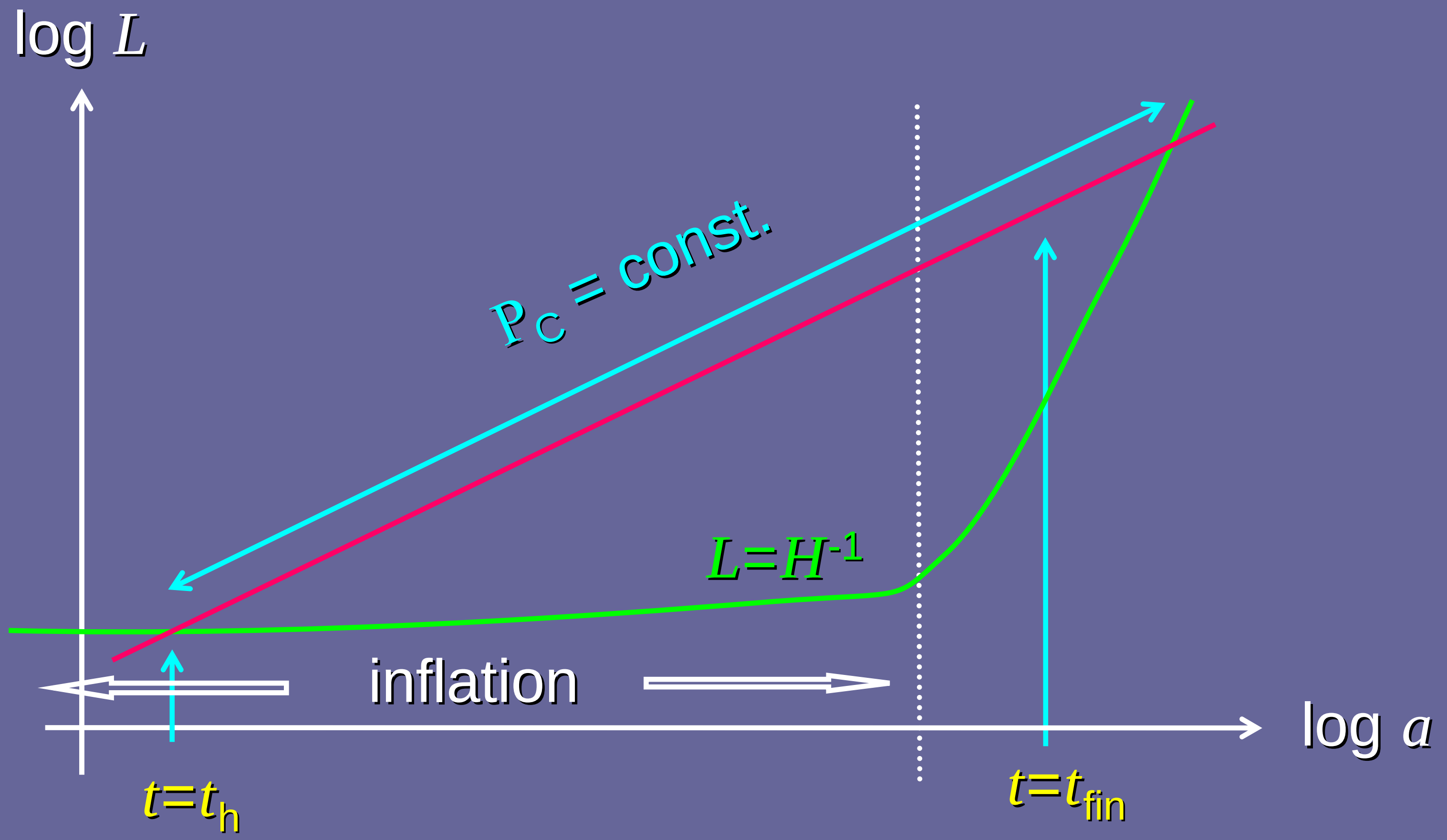
By definition, $\delta N(t;\, t_{\rm fin})$ is t-independent

Example: slow-roll inflation

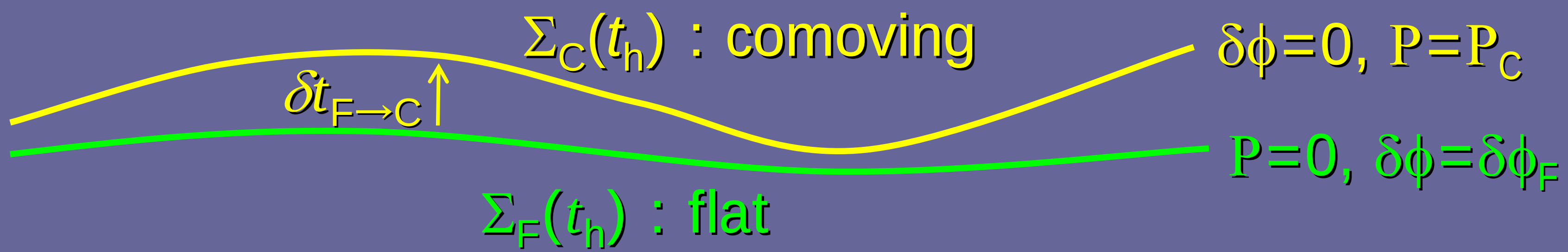
- single-field inflation, no extra degree of freedom

$\mathcal{R}_C$ becomes constant soon after horizon-crossing ($t=t_h$):

$$\delta N\left(t_h;t_{fin}\right)=\mathcal{R}_C\left(t_{fin}\right)=\mathcal{R}_C\left(t_h\right)$$



Also $\delta N = H(t_h)\,\delta t_{F\to C}$, where $\delta t_{F\to C}$ is the time difference between the comoving and flat slices at $t=t_h$.



$$\phi_F\left(t_h+\delta t_{F\to C}, x^i\right)=\phi_C\left(t_h\right) \Rightarrow \delta\phi_F+\dot{\phi}\left(t_h\right)\delta t_{F\to C}=0$$

$$\Rightarrow \mathcal{R}_C\left(t_{fin}\right)=\delta N\left(t_h;t_{fin}\right)=-\frac{H}{d\phi/dt}\delta\phi_F\left(t_h\right) \Leftarrow dN=-Hdt$$

$$=\frac{dN}{d\phi}\delta\phi_F\left(t_h\right)$$

… δN formula

Starobinsky '85

Only the knowledge of the background evolution is necessary to calculate $\mathcal{R}_C(t_{fin})$.

- δN for a multi-component scalar:
  (for slowroll inflation)

$$\mathcal{R}_c(t_{\rm fin}) = \delta N = \sum_a \frac{\partial N}{\partial \phi^a}\delta\phi_{\rm F}^a(t_{\rm h})$$

MS & Stewart '96

N.B. $\mathcal{R}_c$ (=ζ) is no longer constant in time:

$$\mathcal{R}_c(t) = -H\frac{\dot{\phi}\cdot\delta\phi_{\rm F}}{\|\dot{\phi}\|^2}$$

… time varying even on superhorizon scales

$$\left\langle |\mathcal{R}_c|^2(t_{\rm fin})\right\rangle = \|\nabla N\|^2\|\delta\phi_{\rm F}\|^2 = \|\nabla N\|^2\frac{H^2(t_{\rm h})}{(2\pi)^2}$$

$$\nabla_a N \equiv \frac{\partial N}{\partial \phi^a}$$

Further extension to non-slowroll case is possible, if general slow-roll condition is satisfied at horizon-crossing.

Lee, MS, Stewart, Tanaka & Yokoyama '05

$$\frac{\dot{\phi}^2}{2H^2} = O(\xi),\ \frac{\ddot{\phi}}{H\dot{\phi}} = O(\xi),\ \frac{\dddot{\phi}}{H^2\dot{\phi}} = O(\xi),\ \dots,\quad \xi = \ 1$$

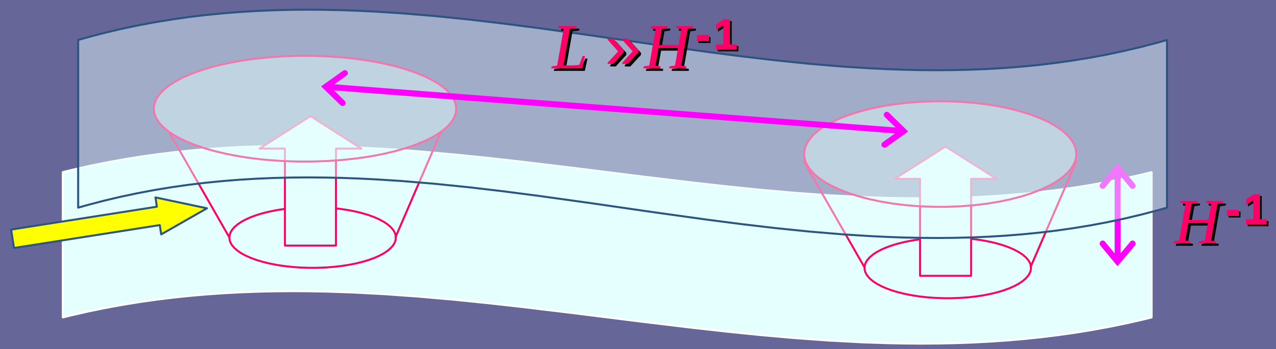
3. Nonlinear extension

- On superhorizon scales, gradient expansion is valid:

$$\left|\frac{\partial}{\partial x^i} Q\right| = \left|\frac{\partial}{\partial t} Q\right| \sim HQ; \quad H \sim \sqrt{G\rho}$$

Belinski et al. '70, Tomita '72, Salopek & Bond '90, ...

This is a consequence of causality:



light cone

- At lowest order, no signal propagates in spatial directions.

Field equations reduce to ODE's

metric on superhorizon scales

- gradient expansion:

$$\partial_i \to \varepsilon \partial_i \,, \quad \varepsilon = \text{expansion parameter}$$

- metric:

$$ds^2 = -N^2 dt^2 + e^{2\alpha} \tilde{\gamma}_{ij} \left(dx^i + \beta^i dt \right)\left(dx^j + \beta^j dt \right)$$

$$\det \tilde{\gamma}_{ij} = 1, \quad \beta^i = O(\varepsilon)$$

the only non-trivial assumption

contains GW (~ tensor) modes

$$\alpha\left(t, x^i\right) = \ln a(t) + \psi\left(t, x^i\right); \ \psi : \ \text{curvature perturbation}$$

e.g., choose $\psi(t_*, 0) = 0$

fiducial `background’

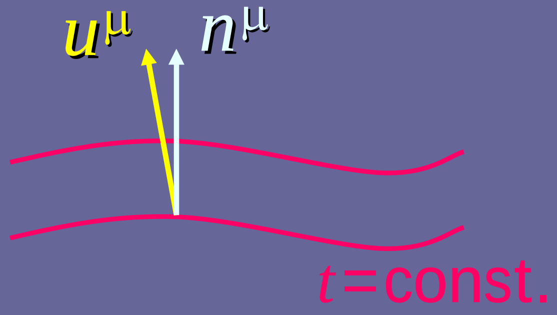
- Energy momentum tensor:

$$T^{\mu\nu} = \rho u^\mu u^\nu + p\left(g^{\mu\nu} + u^\mu u^\nu\right); \quad u_\mu \nabla_\nu T^{\mu\nu} = 0$$

$$\Rightarrow \frac{d}{d\tau}\rho + \nabla_\mu u^\mu (\rho + p) = 0\ ; \ \nabla_\mu u^\mu = 3\frac{\partial_t \alpha}{N} + O\left(\varepsilon^2\right)$$

assumption: $v^i \equiv \frac{u^i}{u^0} = O(\varepsilon)$ ⟹ $u^\mu - n^\mu = O(\varepsilon)$

(absence of vorticity mode)



- Local Hubble parameter:

$$\tilde{H} \equiv \frac{1}{3}\nabla_\mu n^\mu = \frac{1}{3}\nabla_\mu u^\mu + O\left(\varepsilon^2\right)$$

$$n_\mu dx^\mu = -N\,dt \quad \cdots \text{normal to } t = \text{const.}$$

At leading order, local Hubble parameter on any slicing is equivalent to expansion rate of matter flow.

So, hereafter, we redefine $\tilde{H}$ to be $\tilde{H} \equiv \frac{1}{3}\nabla_\mu u^\mu$

- Local Friedmann equation

$$\tilde{H}^2(t,x^i) = \frac{8\pi G}{3}\rho(t,x^i) + O(\varepsilon^2)$$

x^i : comoving (Lagrangean) coordinates.

$$\frac{d}{d\tau}\rho + 3\tilde{H}\left(\rho + p\right) = 0$$

$d\tau =$ ☠ dt : proper time along matter flow

- exactly the same as the background equations.

  “separate universe”

- uniform ρ slice = uniform Hubble slice = comoving slice

  as in the case of linear theory

- no modifications/backreaction due to super-Hubble perturbations.

cf. Hirata & Seljak ‘05
Noh & Hwang ‘05

4. Nonlinear ΔN formula

- energy conservation:

  (applicable to each independent matter component)

$$\frac{\partial_t \rho}{3(\rho + p)} + O(\varepsilon^2) = -\partial_t \alpha = -\left(\frac{\dot{a}}{a} + \partial_t \psi\right) = -\tilde{H}\,\mathrm{N} + O(\varepsilon^2)$$

- e-folding number:

$$N\left(t_2, t_1; x^i\right) \equiv \int_{t_1}^{t_2} \tilde{H}\,\mathrm{N}\, dt = -\frac{1}{3}\int_{t_1}^{t_2} \left.\frac{\partial_t \rho}{\rho + P}\right|_{x^i} dt$$

where x^i=const. is a comoving worldline.

This definition applies to any choice of time-slicing.

⇒ $$\psi\left(t_2, x^i\right) - \psi\left(t_1, x^i\right) = \Delta N\left(t_2, t_1; x^i\right)$$

where

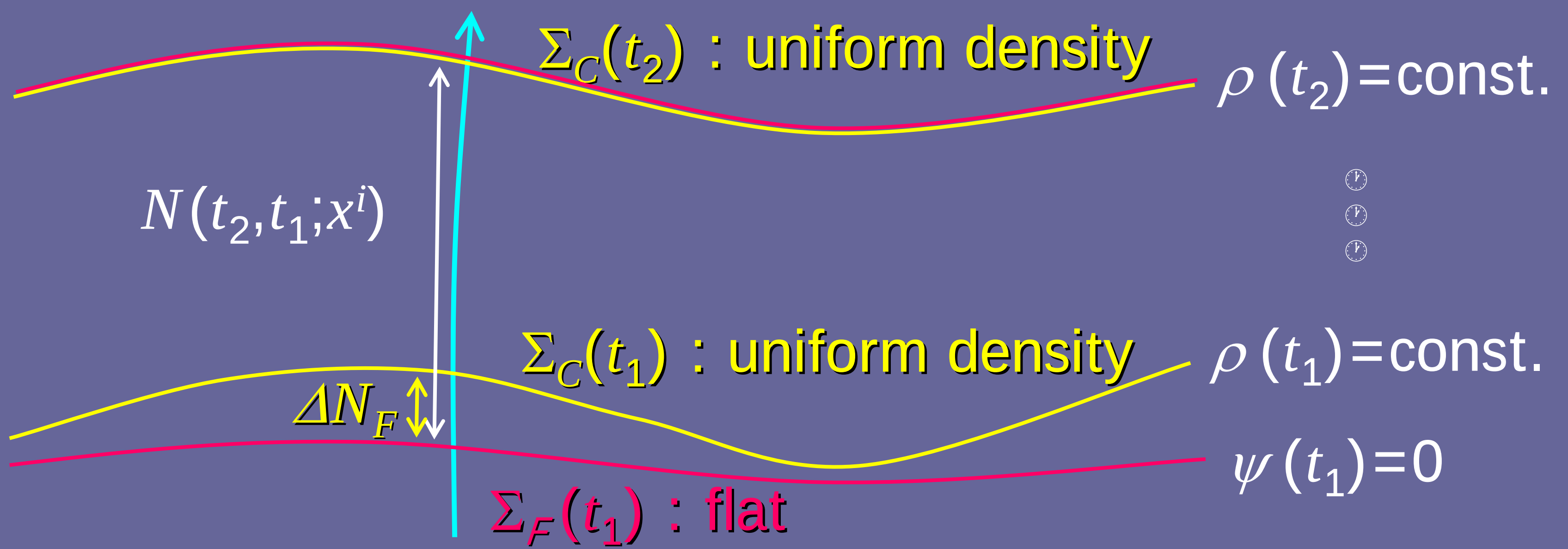
$$\Delta N\left(t_2, t_1; x^i\right) \equiv N\left(t_2, t_1; x^i\right) - \ln\left(\frac{a(t_2)}{a(t_1)}\right)$$

• ΔN - formula

Lyth & Wands '03, Malik, Lyth & MS '04,
Lyth & Rodriguez '05, Langlois & Vernizzi '05

Let us take slicing such that $\Sigma(t)$ is flat at $t = t_1$ [$\Sigma_F(t_1)$]
and uniform density/uniform H/comoving at $t = t_2$ [$\Sigma_C(t_1)$] :

('flat' slice: $\Sigma(t)$ on which $\psi = 0 \leftrightarrow e^{\alpha} = a(t)$)



$$N(t_2, t_1; x^i) = N_0(t_2, t_1) + \Delta N_F$$

$$N_0(t_2, t_1) = \ln\left(\frac{a(t_2)}{a(t_1)}\right) \text{ between } \Sigma_C(t_1) \text{ and } \Sigma_C(t_2)$$

Then

$$\Delta N_F = \psi\left(t_2, x^i\right) - \psi\left(t_1, x^i\right) = \psi_C\left(t_2, x^i\right)$$

suffix C for comoving/uniform ρ/uniform H

where ΔN_F is equal to e-folding number from $\Sigma_F(t_1)$ to $\Sigma_C(t_1)$:

$$\Delta N_F = -\frac{1}{3}\int_{\Sigma_F(t_1)}^{\Sigma_C(t_2)} \left.\frac{\partial_t \rho}{\rho + P}\right|_{x^i} dt + \frac{1}{3}\int_{\Sigma_C(t_1)}^{\Sigma_C(t_2)} \frac{\partial_t \rho}{\rho + P} dt$$

$$= -\frac{1}{3}\int_{\Sigma_F(t_1)}^{\Sigma_C(t_1)} \left.\frac{\partial_t \rho}{\rho + P}\right|_{x^i} dt$$

For slow-roll inflation in linear theory, this reduces to

$$\psi_C(t_2) \equiv \mathcal{R}_C(t_2) = \delta N\left(t_1; t_2\right) = H\left(t_1\right)\delta t_{F\to C} = \left[\sum_a \frac{\partial N}{\partial \phi^a}\delta\phi_F^a\right](t_1)$$

• ΔN for 'slowroll' inflation

MS & Tanaka '98, Lyth & Rodriguez '05

- In slowroll inflation, all decaying mode solutions of the (multi-component) inflaton field ϕ die out.

- If ϕ is slow rolling when the scale of our interest leaves the horizon, N is only a function of ϕ (indep't of $d\phi/dt$, apart from trivial dep. on time t_{fin} from which N is measured), no matter how complicated the subsequent evolution would be.

- Nonlinear ΔN for multi-component inflation :

$$\Delta N = N\left(\phi^A + \delta\phi^A\right) - N\left(\phi^A\right)$$

$$= \sum_n \frac{1}{n!} \frac{\partial^n N}{\partial\phi^{A_1}\partial\phi^{A_2}\cdots\partial\phi^{A_n}} \delta\phi^{A_1}\delta\phi^{A_2}\cdots\delta\phi^{A_n}$$

where $\delta\phi = \delta\phi_F$ (on flat slice) at horizon-crossing.

($\delta\phi_F$ may contain non-gaussianity from subhorizon interactions)

cf. Maldacena '03, Weinberg '05, ...

Diagrammatic method for nonlinear ΔN

Byrnes, Koyama, MS & Wands '07

$$\zeta = \Delta N = \sum_n \frac{N_{A_1 A_2 \cdots A_n}}{n!} \delta\phi^{A_1} \delta\phi^{A_2} \cdots \delta\phi^{A_n} \; ; \qquad N_{A_1 A_2 \cdots A_n} \equiv \frac{D^n N}{\partial\phi^{A_1} \partial\phi^{A_2} \cdots \partial\phi^{A_n}}$$

'basic' 2-pt function: $\left\langle \delta\phi^A(x) \delta\phi^B(y) \right\rangle = h^{AB}(\phi) G_0(x-y)$

field space metric

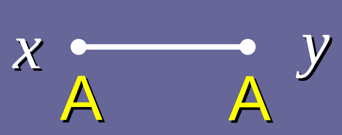
$\delta\phi$ is assumed to be Gaussian

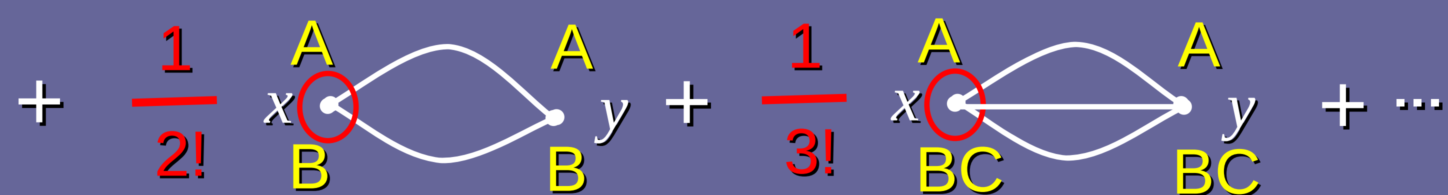
for non-Gaussian $\delta\phi$, there will be basic n-pt functions

- connected n-pt function of ζ:

2-pt function

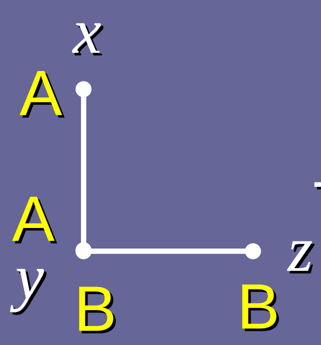
$$\left\langle \zeta(x)\zeta(y) \right\rangle_c = N_A N^A G_0(x-y) + \frac{1}{2!} N_{AB} N^{AB} G_0(x-y)^2 + \frac{1}{3!} N_{ABC} N^{ABC} G_0(x-y)^3 + \cdots$$

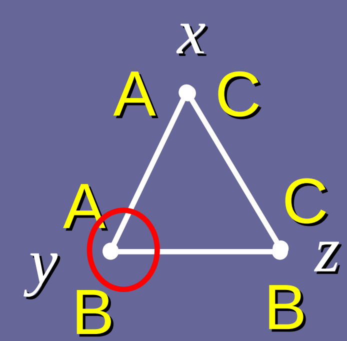
⇔ 

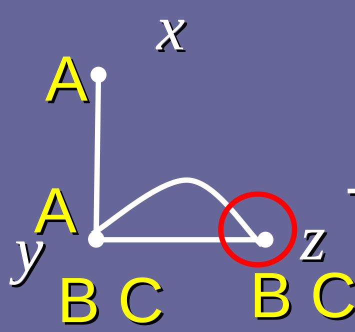


3-pt function

$$\left\langle \zeta(x)\zeta(y)\zeta(z)\right\rangle_c = N^{A} N_{AB} N^{B} G_0(x-y) G_0(y-z) + \text{perm.}$$
$$+ N^{AB} N_{BC} N^{CA} G_0(x-y) G_0(y-z) G_0(z-x)$$
$$+ \frac{1}{2!} N^{A} N_{ABC} N^{BC} G_0(x-y)^2 G_0(y-z) + \text{perm.}$$
$$+ \mathrm{L}$$

⇔ 

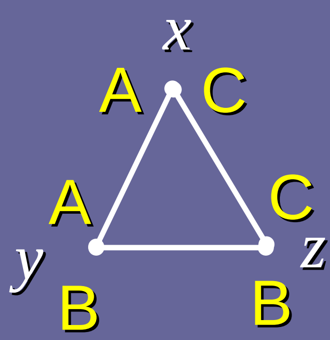
+ perm. + 2! x $\frac{1}{2!}$ 

+ $\frac{1}{2!}$ 

+ perm. + ···

IR divergence problem

Loop diagrams like



$$\Leftrightarrow \quad N^{AB} N_{BC} N^{CA} G_0(x-y) G_0(y-z) G_0(z-x)$$

in the m-pt function give rise to IR divergence in the (m-1)-spectrum if $P(k) \sim k^{n-4}$ with $n \leq 1$.

Boubekeur & Lyth '05

eg, the above diagram gives

$$P(k_1, k_2, k_3) : \ \delta^3(k_1 + k_2 + k_3) \int d^3 p \, P(p) P(|k_1 + p|) P(|k_2 - p|)$$

cutoff-dependent!

Is this IR cutoff physically observable?

(real space 3-pt fcn is perfectly regular if $G_0(x)$ is regular.)

8. Summary

- Superhorizon scale perturbations can never affect local (horizon-size) dynamics, hence never cause backreaction.

  nonlinearity on superhorizon scales are always local.

  However, nonlocal nonlinearity (non-Gaussianity) may appear due to quantum interactions on subhorizon scales.

  cf. Weinberg '06

- There exists a nonlinear generalization of δN formula which is useful in evaluating non-Gaussianity from inflation.

  diagrammatic method can by systematically applied.

  IR divergence from loop diagrams needs further consideration.